THE UNIVERSITY OF ALBERTA

BOUNDS FOR EIGENVALUES

BY

© RAVINDER KUMAR

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL, 1984

ABSTRACT

This thesis deals with localization results for the eigenvalues, their sums and products, the singular values and the condition number of a matrix. Part I of the thesis is in tabular form. Here we give localization results which involve certain pieces of given data. For example, we attempt to answer the following sample question: "Given the data tr A , tr A^2 and that the eigenvalues of the matrix A are real, what more can be said about the localization of the eigenvalues of A ?" The different data used is given in the Table of Contents. We answer these questions only partially, i.e. we do not include all known results. However, it is a step in that direction. In Part II we discuss some specialized topics viz., spectral radius, spread, Gerschgorin disks, singular values and the condition number. Both parts of the thesis also include some interesting theorems about matrices and eigenvalues.

PREFACE

This thesis consists of two parts. Part I is arranged in a tabular form. It consists of chapters one through thirteen which correspond to thirteen different characteristics or properties of a matrix, e.g., $\operatorname{tr} AA^*$, Hermitian, positive definite etc. Each chapter, say chapter t, consists of at most 2^{t-1} sections arranged in lexicographic order. Further, each section contains localization results involving a subset of t characteristics of the first t chapters. For example, if $\operatorname{tr} A$ and $\operatorname{tr} A^2$ of a matrix with real eigenvalues are given, then these pieces of information correspond to chapters 1, 3 and 8, respectively and localization results are given in section 8:3.1. If for certain pieces of data no new results are available, we skip that section. This explains the existence of gaps between sections in the table of contents.

In addition to localization results, the first section of each chapter also contains some basic definitions and / or some useful results. Each subsequent section has results only under the headings (a) through (h), whenever available. Each of these headings contains some specific results as indicated below:

(a) For the spectral radius or the largest eigenvalue.

(b) For the smallest eigenvalue.

(c) For the kth eigenvalue.

(d) For the spread.

(e) For sum of eigenvalues.

(f) For product of eigenvalues.

(g) For singular values.

(h) For the condition number.

In Part II we discuss some specialized topics, namely, spectral radius, spread, Gerschgorin disks, singular values and the condition number. In each chapter we give some theorems and localization results.

ACKNOWLEDGEMENTS

I am indebted to my supervisor, Professor Henry Wolkowicz, for his help and encouragement during the preparation of this thesis. He originally suggested the topic of the thesis and continued to advise generously throughout this work.

I would also like to express my thankfulness to my family and friends for their support over the years.

TABLE OF CONTENTS

Page

CHAPTER

INTRODUCTION 1

PART I

1 TRACE A

1:0 Preliminaries 10

2 DIAGONAL ELEMENTS

2:0 Preliminaries 13

3 TRACE A^2

3:0 Preliminaries 17

3:1 Trace A and Trace A^2 19

3:2 Diagonal Elements and Trace A^2 20

4 TRACE AA^*

4:0 Preliminaries 21

4:1 Trace A and Trace AA^* 25

4:2 Diagonal Elements and Trace AA^* 28

4:3 Trace A^2 and Trace AA^* 29

4:3.1 Trace A, Trace A^2 and Trace AA^* 31

5 TRACE $(AA^* - A^*A)^2$

5:0 Preliminaries 35

5:4 Trace AA^* and Trace $(AA^* - A^*A)^2$ 37

5:4.1 Trace A, Trace AA^* and Trace $(AA^* - A^*A)^2$ . . . 38

5:4.3 Trace A^2, Trace AA^* and Trace $(AA^* - A^*A)^2$ . . 43

5:4.3.1 Trace A, Trace A^2, Trace AA^*, Trace $(AA^* - A^*A)^2$. 44

6 ROW AND COLUMN SUMS

6:0 Preliminaries 47
6:2 Diagonal Elements and Row and Column Sums 54
6:4 Trace AA^* and Row and Column Sums 62
6:5 Trace $(AA^* - A^*A)^2$ and Row and Column Sums. . . . 64

7 DETERMINANT

7:0 Preliminaries 65
7:1 Trace A and Determinant 69
7:2 Diagonal Elements and Determinant 70
7:3 Trace A^2 and Determinant. 71
7:4 Trace AA^* and Determinant 72
7:5 Trace $(AA^* - A^*A)^2$ and Determinant. 76
7:5.4 Trace AA^*, Trace $(AA^* - A^*A)^2$ and Determinant . . 77
7:6 Row and Column Sums and Determinant 79

8 REAL EIGENVALUES

8:0 Preliminaries 80
8:1 Trace A and Real Eigenvalues. 82
8:3 Trace A^2 and Real Eigenvalues 83
8:3.1 Trace A, Trace A^2 and Real Eigenvalues. 85
8:4 Trace AA^* and Real Eigenvalues. 95

9 REAL MATRIX

9:0 Preliminaries 96
9:4.2 Diagonal Elements, tr AA^* and Real Matrix 97
9:6 Row and Column Sums and Real Matrix 99
9:6.2 Diagonal Elements, Row and Column Sums and Real Matrix. 101

10 NONNEGATIVE MATRIX

10:0 Preliminaries 103
10:1 Trace A and Nonnegative Matrix. 110
10:2 Diagonal Elements and Nonnegative Matrix. 112
10:3 Trace A^2 and Nonnegative Matrix 113
10:5.4.3 Trace A^2, Trace AA^*, Trace $(AA^* - A^*A)^2$ and Nonnegative Matrix. 115

10:6 Row and Column Sums and Nonnegative Matrix 117
10:6.2 Diagonal Elements, Row and Column Sums and Nonnegative Matrix 120
10:7 Determinant and Nonnegative Matrix 121
10:7.6 Row and Column Sums, Determinant and Nonnegative Matrix 122
10:8 Real Eigenvalues and Nonnegative Matrix 123

11 NORMAL MATRIX

11:0 Preliminaries 124
11:2 Diagonal Elements and Normal Matrix 128
11:4 Trace AA^* and Normal Matrix 130
11:4.3 Trace A^2, Trace AA^* and Normal Matrix 131
11:4.3.1 Trace A, Trace A^2, Trace AA^* and Normal Matrix. . . 132
11:6 Row and Column Sums and Normal Matrix 135
11:8 Real Eigenvalues and Normal Matrix. 138
11:9.6.1 Trace A, Row and Column Sums and Real, Normal Matrix 140

12 HERMITIAN MATRIX

12:0 Preliminaries 141
12:2 Diagonal Elements and Hermitian Matrix. 146
12:3.1 Trace A, Trace A^2 and Hermitian Matrix. 149
12:6 Row Sums and Hermitian Matrix 151
12:6.2 Row Sums, Diagonal Elements and Hermitian Matrix. . 153
12:10.2 Nonnegative Matrix, Diagonal Elements and Hermitian Matrix. 154

13 POSITIVE DEFINITE (SEMIDEFINITE) MATRIX
13:0 Preliminaries 155
13:1 Trace A Positive definite (semidefinite) Matrix . . 159
13:2 Diagonal Elements and Positive definite (semidefinite) Matrix 160
13:3.1 Trace A, Trace A^2 and Positive definite (semidefinite) Matrix 162
13:3.2 Diagonal Elements, Trace A^2 and Positive definite (semidefinite) Matrix 163
13:6 Row Sums and Positive definite (semidefinite) Matrix 166

13:6:2 Diagonal Elements, Row Sums and Positive definite (semidefinite) Matrix 167
13:7.1 Trace A, Determinant and Positive definite (semidefinite) Matrix 168
13:7.2 Diagonal Elements, Determinant and Positive definite (semidefinite) Matrix. 170
13:7.3.1 Trace A, Trace A^2, Determinant and Positive definite (semidefinite) Matrix. 171

PART II

1 SPECTRAL RADIUS

1:0 Preliminaries 173
1:1 Matrix orms 174
1:2 Bounds for Spectral Radius. 179

2 SPREAD

2:0 Preliminaries 183
2:1 Bounds for Spread 186

3 GERSCHGORIN DISKS

3:0 Preliminaries 191
3:1 Localization Results. 195
3:2 Real Matrix 200

4 SINGULAR VALUES

4:0 Preliminaries 201
4:1 Singular Values and Eigenvalues 204
4:2 The Matrices A, B and C 206
4:3 The Matrices AA^*, B and C 208
4:4 Bounds for Singular Values. 210
4:5 Bounds for Real and Imaginary Singular Values . . . 217

5 CONDITION NUMBER

5:0 Preliminaries 218
5:1 Localization Results. 222

BIBLIOGRAPHY 224

THE NUMBERING SYSTEM AND SYMBOLS

Within the text of a chapter a reference to a section in that chapter will only give the number of the section. A reference to a section in some other chapter, but in the same part, will be preceded by the number of the chapter and will be separated by a colon. In case a section of a different part is referenced, the chapter number will be preceded by the part number and will be separated by a colon. For example, if we wish to refer to section 2 of chapter 4 of Part II in Part I, we write "II:4:2", while referring to it in a different chapter of Part II we write "4:2". Equations and theorems are referred to by the section number followed by the equation number or the theorem number in parentheses. Thus if we refer to Theorem 1 of Section 1 and Chapter 4 of Part II, in Part I, we write 'Theorem II:4:1 (1)', while referring to it in a different chapter of Part II we write "Theorem 4:1 (1)". External references are always given in closed brackets. In case a book is referred to, the reference number is followed by the appropriate page number.

In addition, we shall use the following symbols:

(1) $!$ - factorial

(2) i - $\sqrt{-1}$

(3) $|u|$ - absolute value of the complex number u .

(4) $\mathrm{Re}(u)$ - real part of the complex number u .

(5) $\mathrm{Im}(u)$ - imaginary part of the complex number u .

(6) $¢^m$ - set of all complex $m \times 1$ vectors.

(7) e_i - a vector with 1 as the ith component and zeros elsewhere.

(8) I - the identity matrix of order n .

(9) A' - transpose of A .

(10) A^* - conjugate transpose of A .

(11) B - $\frac{1}{2}(A+A^*)$.

(12) C - $\frac{1}{2i}(A-A^*)$.

(13) $\mathrm{diag}(x_1,x_2,\cdots,x_n)$ - a diagonal matrix with x_i as the ith diagonal element.

(14) $A > (\geq) 0$ - elements of A are positive (nonnegative).

(15) (x,y) - x^*y , the inner product of x and y .

(16) $||x||^2$ - x^*x .

(17) $x \prec y$ - y majorizes x .

(18) $x \prec_w y$ - y majorizes x , weakly.

(19) λ_i - an eigenvalue of A .

(20) μ_i - an eigenvalue of B .

(21) ν_i - an eigenvalue of C .

(22) σ_i - a singular value of A .

(23) $\mathrm{tr}\, A$ - trace of A .

(24) $\mathrm{sp}(A)$ - $\max_{i,j} |\lambda_i-\lambda_j|$, the spread of A .

(25) $\mathrm{sp}_R(A)$ - $\max_{i,j} (\mathrm{Re}(\lambda_i) - \mathrm{Re}(\lambda_j))$.

(26) $\mathrm{sp}_I(A)$ - $\max_{i,j} (\mathrm{Im}(\lambda_i) - \mathrm{Im}(\lambda_j))$.

(27) $\det A$ - determinant of A .

(28) $c(A)$ - the condition number of A .

(29) $\rho(A)$ - spectral radius of A .

(30) $\rho(x)$ - $(x,Ax)/(x,x)$.

INTRODUCTION

Given a complex matrix $A = (a_{ij})$ of order n, a complex scalar λ is called an eigenvalue of A if there exists a non-zero vector x such that $(A-\lambda I)x = 0$. Thus λ is an eigenvalue of A if and only if $\det(A-\lambda I)$, called the characteristic polynomial of A, is zero. Since the degree of the characteristic polynomial of A is n, we conclude that a matrix of order n has exactly, counting multiplicities, n eigenvalues. We shall always denote the eigenvalues of A by λ_k, $k = 1,2,\cdots,n$. We also note that the positive square roots of the eigenvalues of the matrix AA^*, where A^* is the conjugate transpose of A, are called the singular values of A.

In many applications the approximate location of the eigenvalues of a matrix suffices and, also, because of the effort involved in computing the eigenvalues of A, it becomes expedient to approximate them by some means. For example, in many iterative methods for solving a system of linear equations the method converges if the spectral radius of a certain matrix is less than one. In the theory of differential equations the linear homogenous system $\frac{dx}{dt} = Ax$ is said to have a asymtotically stable solution if all the eigenvalues of A have negative real parts. Further, the condition number, $c(A) = ||A||\ ||A^{-1}||$, where $||\cdot||$ is a matrix norm, is useful for determining whether or not the system of linear equations $Ax = b$ is well-conditioned. In optimization the condition number of the Hessian of a certain matrix at the solution provides a measure of the sensitivity of the optimal solution. In case of the 2-norm or the spectral norm, when A is nonsingular, $c(A) = \sigma_1 / \sigma_n$, the

ratio of the largest and smallest singular values. Thus, an estimate for σ_1 and σ_n of a nonsingular matrix provides an estimate for $c(A)$. Further, as we shall see in Chapter 5 of Part II, for certain matrix norms we get $\max_i |\lambda_i| / \min_i |\lambda_i| \leq c(A)$. Therefore, bounds for $\max_i |\lambda_i|$ and $\min_i |\lambda_i|$ can be used for estimating $c(A)$.

In this thesis we shall be concerned with the localization of the eigenvalues, their sums and products, the spectral radius, the spread, the singular values and the condition number of a matrix.

Localization results abound in the literature. Marcus and Minc in their book (see Chapter III of [31]) give a brief history of these results. For example, in 1909, Issai Schur proved the following:

$$\sum_i |\lambda_i|^2 \leq \operatorname{tr} AA^* \quad , \tag{1}$$

$$\sum_i \operatorname{Re}(\lambda_i)^2 \leq \operatorname{tr} B^2 = \frac{1}{2}\left(\operatorname{tr} AA^* + \operatorname{Re}(\operatorname{tr} A^2)\right) \quad , \tag{2}$$

$$\sum_i \operatorname{Im}(\lambda_i)^2 \leq \operatorname{tr} C^2 = \frac{1}{2}\left(\operatorname{tr} AA^* - \operatorname{Re}(\operatorname{tr} A^2)\right) \quad , \tag{3}$$

with equality if and only if A is normal. In 1946, Alfred Brauer proved that, for an arbitrary matrix A,

$$\max_i |\lambda_i| \leq \min\,(R,C) \quad , \tag{4}$$

where

$$R_i = \sum_j |a_{ij}| \quad , \qquad C_i = \sum_j |a_{ji}| \quad ,$$

$$R = \max_i R_i \quad \text{and} \quad C = \max_i C_i \quad .$$

However, this result was anticipated by Oskar Perron in 1933 and also

follows from Gerschgorin's Theorem, which was proved in 1931 (see [31, pg. 145]). Gerschgorin proved that all the eigenvalues of a matrix A lie in the disks

$$|z - a_{kk}| \leq R_k - |a_{kk}| \quad , \quad k = 1,2,\cdots,n \quad . \tag{5}$$

The proof of the above follows readily from the Lev̌y-Desplanques Theorem, proved in 1887. Another well-known result is the Frobenius Theorem for nonnegative matrices. It states that if A is nonnegative, i.e. $a_{ij} \geq 0$ for all i and j, then

$$\min_i R_i \leq \max_i |\lambda_i| \leq \max_i R_i \quad . \tag{6}$$

The theory of nonnegative matrices, originated by Perron and Frobenious, has proved to be very useful (see e.g. [51] and [59]).

More recently Henry Wolkowicz and George P.H. Styan obtained several localization results using traces (see [63] and [64]). For example, they showed

$$|m| - s_a\left(\frac{k-1}{n-k+1}\right)^{1/2} \leq |\lambda_k| \leq \left(\frac{\operatorname{tr} AA^*}{n}\right)^{1/2} + s_a\left(\frac{n-k}{k}\right)^{1/2} \quad , \quad k=1,2,\cdots,n \tag{7}$$

where, $m = \operatorname{tr} A / n$ and $s_a{}^2 = \operatorname{tr} AA^* / n - |m|^2$.

All the above mentioned results are included in Part I. Also included are many other results. For example, in Chapter 2 of Part I we include the inequalities:

$$\sum_1^k d_i \leq \sum_1^k \sigma_i \quad , \quad k = 1,2,\cdots,n \quad ,$$

where d_i's are the moduli of the diagonal elements, arranged in decreasing order. Chapter 5 includes improved versions of inequalities (1), (2) and (3). In Chapter 6 we give many localization results involving row and column sums, including inequalities (4) and (5). Also included are several conditions which ensure the nonsingularity of a matrix. For example, it is shown that, $\det A \neq 0$ if

$$|a_{ii}| > R_i - |a_{ii}| \quad , \quad i = 1,2,\cdots,n \quad .$$

In Chapter 7 bounds for eigenvalues are derived by means of the arithmetic-geometric mean inequality. For example, we prove that

$$\left(\frac{n-1}{\operatorname{tr} AA^*}\right)^{n-1} |\det A|^2 \leq |\lambda_n|^2 \leq \operatorname{tr} AA^* - (n-1)\, |\det A|^{2/n} \quad .$$

In Chapter 8 we give necessary and sufficient conditions for a matrix to have real eigenvalues. Also included are several eigenvalue bounds involving $\operatorname{tr} A$ and $\operatorname{tr} A^2$. For example,

$$m - s\left(\frac{k-1}{n-k+1}\right)^{1/2} \leq \lambda_k \leq m + s\left(\frac{n-k}{k}\right)^{1/2} \quad , \quad k = 1,2,\cdots,n \quad ,$$

where

$$m = \operatorname{tr} A / n \qquad \text{and} \qquad s^2 = \operatorname{tr} A^2 / n - m^2 \quad ,$$

and

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad .$$

Chapter 10 deals with nonnegative matrices. Here, we first classify the nonnegative matrices and then state their interesting spectral properties. Chapters 11, 12 and 13 deal with a special class of matrices, the normal

matrices. Such matrices are always unitarily diagonalizable. Also, for a normal matrix the moduli of the eigenvalues, $|\lambda_i|$, $i = 1,2,\cdots,n$, are its singular values. Thus all the results for singular values in previous chapters hold for the eigenvalues.

In Part II we consider some specialized topics. In Chapter 1 we discuss the spectral radius. In particular we show that for certain matrix norms,

$$\max_i |\lambda_i| = \rho(A) \leq ||A|| \quad .$$

Also included are some results which improve upon inequality (6) above. In Chapter 2 we give results for the spread of a matrix. In Chapter 3 we briefly discuss Gerschgorin's disks. In Chapter 4 we give several inequalities relating the singular values with the eigenvalues, real singular values and imaginary singular values. For example,

$$\prod_{i=1}^{k} |\lambda_i| \leq \prod_{i=1}^{k} \sigma_i \quad , \quad k = 1,2,\cdots,n \quad ,$$

$$\sigma_n \leq |\lambda_n| \quad ,$$

$$\mu_k \leq \sigma_k \quad , \quad k = 1,2,\cdots,n \quad ,$$

and

$$\nu_k \leq \sigma_k \quad , \quad k = 1,2,\cdots,n \quad ,$$

where, σ_k , μ_k and ν_k are the singular values, real singular values and imaginary singular values of A , respectively. Also included are bounds for the singular values themselves. For example,

$$\max_{1 \le s,t \le n-1} \frac{1}{\sqrt{st}} \left| \sum_{j=1}^{t} \sum_{i=1}^{s} a_{ij} \right| \le \sigma_1 \quad ,$$

$$\sigma_n \le \min_i R_i = \min_i \sum_j |a_{ij}| \quad .$$

In Chapter 5 we briefly discuss the condition number.

Now we shall give some definitions which are assumed throughout the thesis. First, let $A = (a_{ij})$ be an $m \times n$ complex matrix. Unless otherwise stated we shall always assume that $m = n$, that is A is a square matrix of order n. Let λ_i, $i = 1,2,\cdots,n$ be the eigenvalues of A. Generally, we shall assume that λ_i, $i = 1,2,\cdots,n$ are ordered as $|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_n|$, if complex and as $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, if real.

Definition 1: A matrix P is called a permutation matrix if each row and each column of P has some one entry unity and all others zero. □

Definition 2: For $n \ge 2$ a complex matrix A of order n is called reducible if there exists a permutation matrix P such that

$$PAP' = \begin{pmatrix} D & F \\ 0 & E \end{pmatrix} ,$$

where D and E are matrices of order $r < n$ and s respectively such that $r+s = n$. □

Definition 3: If a matrix A is not reducible then it is called irreducible. □

Definition 4: A matrix A is called skew-Hermitian if $A^* = -A$. □

Definition 5: Given a matrix A , it is called a scalar matrix, if $A = \alpha I$, for some complex number α . □

Definition 6: A matrix A , is called nonnegative, if $a_{ij} \geq 0$, $1 \leq i , j \leq n$. In case $a_{ij} > 0$, $1 \leq i , j \leq n$, we say A is positive. □

Definition 7: A matrix A is called upper triangular if

$$i > j \Rightarrow a_{ij} = 0 ,$$

and lower triangular if

$$i < j \Rightarrow a_{ij} = 0 .$$

It is called a diagonal matrix if $a_{ij} = 0$ for $i \neq j$. □

Definition 8: A matrix U is called unitary if $UU^* = U^*U = I$. □

Definition 9: If D and E are two matrices of order $n \times n$ then they are called similar if there exists a nonsingular matrix S such that $D = SES^{-1}$. In case S is unitary we say D and E are unitarily similar. □

Definition 10: Let A be any $m \times n$ matrix and let $1 \leq i_1 < i_2 < \cdots < i_k \leq m$ and $1 \leq j_1 < j_2 < \cdots < j_\ell \leq n$. The $k \times \ell$ matrix S whose (α,β)th element is

$$s_{\alpha\beta} = a_{i_\alpha j_\beta} \quad ,$$

is called a <u>submatrix</u> of A. In case $k = \ell$ and $i_1 = j_1, i_2 = j_2, \cdots, i_k = j_k$, then S is called a <u>principal</u> <u>submatrix</u> of A. If $i_1 = 1$, $i_2 = 2$, $\cdots$, $i_k = k$ and $j_1 = 1$, $j_2 = 2$, $\cdots$, $j_\ell = \ell$, then S is called a <u>leading</u> <u>submatrix</u> of A. If a submatrix is both principal and leading then it is called a <u>leading</u> <u>principal</u> submatrix. □

<u>Definition 11</u>: Let S be a submatrix of A of order $r \leq n$. Then the determinant of S is called a <u>minor</u> of A of order r. In case S is a principal submatrix of A, the determinant of S is called a <u>princi-pal</u> <u>minor</u> of A of order r. Finally, if S is a leading principal submatrix then determinant of S is called a <u>leading</u> <u>principal</u> <u>minor</u> of A.

□

<u>Definition 12</u>: Let x and y be any two real vectors. Let the indices $i_1, i_2, \cdots, i_n$ be such that

$$x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_n} \quad ,$$

and

$$y_{i_1} \geq y_{i_2} \geq \cdots \geq y_{i_n} \quad .$$

If

$$\sum_{j=1}^{k} x_{i_j} \leq \sum_{j=1}^{k} y_{i_j} \quad , \quad k = 1,2,\cdots,n \quad ,$$

then we say y <u>weakly</u> <u>majorizes</u> x, and write $x \prec_w y$. If in addition $\sum_i x_i = \sum_i y_i$, we say y <u>majorizes</u> x and write $x \prec y$. □

PART I

TABLE OF BOUNDS USING GIVEN DATA

CHAPTER 1

TRACE A

§1:0 Preliminaries.

The trace of $A = (a_{ij})_{n\times n}$, denoted by $\operatorname{tr} A$, is defined as:

$$\operatorname{tr} A = \sum_i a_{ii} \quad . \tag{1}$$

For any two matrices H and K, (1) implies that

$$\operatorname{tr} HK = \operatorname{tr} KH \quad . \tag{2}$$

It follows from (2) that similar matrices have the same trace. Using this fact together with Schur's triangularization theorem, the following result follows:

Theorem 1: For any matrix A and positive integer k,

$$\operatorname{tr} A^k = \sum_i \lambda_i^k \quad . \tag{3}$$

Proof: From Schur's triangularization theorem (see e.g. [31, pg. 67]) there exists a unitary matrix U such that $T = U^*AU$ is an upper triangular matrix with the eigenvalues of A along the diagonal. Thus,

$$A^k = (UTU^*)^k = UT^kU^* \quad ,$$

and we have,

$$\mathrm{tr}\ A^k = \mathrm{tr}(UT^kU^*) = \mathrm{tr}\ T^k = \sum_i \lambda_i^k \quad ,$$

which completes the proof.

□

Now we shall derive bounds for the eigenvalues and the singular values of A which involve only $\mathrm{tr}\ A$ and n .

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

Given $\mathrm{tr}\ A$,

$$|\mathrm{tr}\ A| \,/\, n \leq |\lambda_1| \quad . \tag{4}$$

Equality holds if and only if,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n \quad .$$

<u>Proof</u>: From Theorem (1), we have $n|\lambda_1| \geq \sum_i |\lambda_i| \geq |\mathrm{tr}\ A|$, which proves (4). The conditions for equality are clear.

□

The following result is immediate from (3):

(e) For sum of eigenvalues:

Given $\mathrm{tr}\ A$,

$$\sum_i \lambda_i = \mathrm{tr}\ A \quad ; \tag{5}$$

and

$$|\mathrm{tr}\, A| \leq \sum_i |\lambda_i| \quad . \tag{6}$$

Equality holds in (6) if and only if $\lambda_j = \alpha_j \lambda_1$ for some nonnegative α_j , $j = 2,3,\cdots,n$.

□

(g) For singular values:

Given $\mathrm{tr}\, A$,

$$\frac{|\mathrm{tr}\, A|}{n} \leq |\lambda_1| \leq \sigma_1 \quad ; \tag{7}$$

$$|\mathrm{tr}\, A| \leq \sum_i |\lambda_i| \leq \sum_i \sigma_i \quad . \tag{8}$$

For normal A , equality holds in (7) if and only if A is a scalar matrix, and equality holds in (8) if and only if $\lambda_j = \alpha_j \lambda_1$ for some nonnegative α_j , $j = 2,3,\cdots,n$.

<u>Proof</u>: From Theorem II:4:1 (1), $\sum_1^k |\lambda_i| \leq \sum_1^k \sigma_i$, $1 \leq k \leq n$. Thus inequalities (7) and (8) are immediate from (4) and (6). The conditions for equality are clear from Theorem II:0 (7) and Theorem II:4:1 (2).

□

CHAPTER 2

DIAGONAL ELEMENTS

§2:0 Preliminaries.

The diagonal elements of a matrix provide quite useful information about its eigenvalues. For example, if a matrix A is diagonally dominant, that is

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad , \quad i = 1,2,\cdots,n \quad ,$$

then $\det A$ is non-zero (see 6:2). Also, the eigenvalues of an Hermitian matrix A majorize its diagonal elements (see 12:2 (6)).

We shall often assume that the diagonal elements, a_{ii} $i = 1,2,\cdots,n$, of A are ordered as:

$$|d_1| \geq |d_2| \geq \cdots \geq |d_n| \quad , \tag{1}$$

where $d_i = a_{jj}$ for some $1 \leq j \leq n$, $i = 1,2,\cdots,n$.

Next, we state the necessary and sufficient conditions for the existence of a matrix with prescribed diagonal elements and eigenvalues or singular values. The proofs can be found in [32, pg. 230] and [56], respectively.

Theorem 1: A necessary and sufficient condition for the existence of a real (complex) matrix A with real (complex) eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$

and diagonal elements a_{ii} , $i = 1,2,\cdots,n$ is:

$$\sum_i a_{ii} = \sum_i \lambda_i \quad . \tag{2}$$

$\square$

Theorem 2: The necessary and sufficient conditions for the existence of an $n \times n$ matrix with diagonal elements a_{ii} , $i = 1,2,\cdots,n$ and singular values, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ are:

$$(|d_1|,|d_2|,\cdots,|d_n|) <_w (\sigma_1,\sigma_2,\cdots,\sigma_n) \quad ; \tag{3}$$

$$\sum_1^{n-1} |d_i| - |d_n| \leq \sum_1^{n-1} \sigma_i - \sigma_n \quad , \tag{4}$$

where the $|d_i|$'s are given by (1).

$\square$

Next, we give bounds for the eigenvalues and singular values of A , which involve only the diagonal elements of A and n .

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

Given a_{ii} , $i = 1,2,\cdots,n$,

$$\frac{1}{n} \left|\sum_i a_{ii}\right| \leq |\lambda_1| \quad . \tag{5}$$

Equality holds if and only if,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n \quad .$$

Proof: Since $\operatorname{tr} A = \sum_i a_{ii}$, the result is immediate from 1:0 (4).

□

(e) For sum of eigenvalues:

Given a_{ii}, $i = 1,2,\cdots,n$,

$$\left|\sum_i a_{ii}\right| \leq \sum_i |\lambda_i| \quad . \tag{6}$$

Equality holds if and only if $\lambda_j = \alpha_j \lambda_1$ for some nonnegative α_j, $j = 2,3,\cdots,n$.

Proof: By definition $\operatorname{tr} A = \sum_i a_{ii}$. Now (6) and the equality conditions follow from 1:0 (6).

□

(g) For singular values:

Let the diagonal be ordered as in (1). Then:

$$\sum_1^k |d_i| \leq \sum_1^k \sigma_i \quad , \quad k = 1,2,\cdots,n \quad ; \tag{7}$$

$$\sum_1^{n-1} |d_i| - |d_n| \leq \sum_1^{n-1} \sigma_i - \sigma_n \quad . \tag{8}$$

Equality holds in (7) and (8) if and only if A is diagonal.

Proof: Inequalities (7) and (8) are clear from Theorem (2). Further if equality holds in (7) we have $\sigma_i = |d_i|$,

$i = 1,\cdots,n$. Thus, $\operatorname{tr} AA^* = \sum_i \sigma_i^{\,2} = \sum_{i,j} |a_{ij}|^2 = \sum_i |a_{ii}|^2$, that is all the off-diagonal elements of A are zero. The converse is clear.

$\square$

CHAPTER 3

TRACE A^2

§3:0 Preliminaries.

By definition of the trace of a matrix,

$$\operatorname{tr} A^2 = \sum_{i,j} a_{ij}\, a_{ji} \quad . \tag{1}$$

Below, we derive bounds for the eigenvalues and the singular values, which involve only $\operatorname{tr} A^2$ and n .

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

Given $\operatorname{tr} A^2$,

$$\left(\frac{|\operatorname{tr} A^2|}{n}\right)^{1/2} \leq |\lambda_1| \quad . \tag{2}$$

Equality holds if and only if, $\lambda_1 = \lambda_2 = \cdots = \lambda_n$.

Proof: Inequality (2) is immediate from the triangular inequality and 1:0 (3). Also, the conditions for equality are clear. □

(e) For sum of eigenvalues:

The following result follows from 1:0 (3):

Given $\operatorname{tr} A^2$,

$$\sum_i \lambda_i^2 = \operatorname{tr} A^2 \quad ; \tag{3}$$

$$|\operatorname{tr} A^2| \le \sum_i |\lambda_i^2| \ . \tag{4}$$

Equality holds in (4) if and only if $\lambda_j = \alpha_j \lambda_1$, for some nonegative α_j , $j = 1,2,\cdots,n$.

□

(g) For singular values:

Given $\operatorname{tr} A^2$,

$$\left(\frac{|\operatorname{tr} A^2|}{n}\right)^{1/2} \le \sigma_1 \quad ; \tag{5}$$

and

$$|\operatorname{tr} A^2| \le \sum_i \sigma_i^2 \ . \tag{6}$$

For normal A , equality holds in (5) if and only if A is a scalar matrix and in (6) if and only if $\lambda_j = \alpha_j \lambda_1$ for some nonnegative α_j , $j = 1,2,\cdots,n$.

Proof: Inequalities (5) and (6) follow from (2) and (4) and Theorem II:4:1 (1). The equality conditions are clear from Theorems II:4:1 (2) and 11:0 (7).

□

§3:1

Given $\operatorname{tr} A$ and $\operatorname{tr} A^2$ the results of Section 0 and 1:0 can be combined to yield the following:

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

Given $\operatorname{tr} A$ and $\operatorname{tr} A^2$,

$$\max\left(\frac{|\operatorname{tr} A|}{n}, \frac{|\operatorname{tr} A^2|}{n}^{1/2}\right) \leq |\lambda_1| . \tag{1}$$

Equality holds if and only if all the eigenvalues are equal.

□

(g) For singular values:

Given $\operatorname{tr} A$ and $\operatorname{tr} A^2$,

$$\max\left(\frac{|\operatorname{tr} A|}{n}, \frac{|\operatorname{tr} A^2|}{n}^{1/2}\right) \leq \sigma_1 . \tag{2}$$

□

§3:2

In addition to results of Section 1, the following result holds:

(g) For singular values:

Given $\operatorname{tr} A^2$ and the diagonal elements of A,

$$\max\left(\left|\frac{\operatorname{tr} A^2}{n}\right|^{1/2}, \max_i |a_{ii}|\right) \leq \sigma_1 \quad . \tag{1}$$

<u>Proof</u>: Inequality (1) is immediate from 0 (5) and 2:0 (7). $\square$

CHAPTER 4

TRACE AA^*

§4:0 Preliminaries.

Given an $n \times n$ matrix A, by definition $\operatorname{tr} AA^* = \sum_{i,j} |a_{ij}|^2$. Also, the Euclidean norm (Frobenius norm) of A is defined as:

$$\|A\|^2 = \operatorname{tr} AA^* \quad . \tag{1}$$

Further, AA^* is positive semidefinite, as $(AA^*)^* = AA^*$ and $(AA^*x,x) = (A^*x,A^*x) \geq 0$. Thus, the eigenvalues of AA^* are real, non-negative. The positive square roots of the eigenvalues of AA^* are called the singular values of A. The relationships among the eigenvalues and the singular values of a matrix are discussed in II:4:1.

As always, we shall assume that the singular values of A are ordered as:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \quad . \tag{2}$$

It follows from the above discussion that,

$$\sum_i \sigma_i^2 = \operatorname{tr} AA^* \quad . \tag{3}$$

Next, we give bounds which involve only $\operatorname{tr} AA^*$ and n.

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

Given $\operatorname{tr} AA^*$,

$$|\lambda_1|^2 \leq \operatorname{tr} AA^* . \tag{4}$$

Proof: Inequality (4) is clear from (9) below.

□

(b) For $|\lambda_n| = \min_i |\lambda_i|$:

Given $\operatorname{tr} AA^*$,

$$|\lambda_n|^2 \leq \operatorname{tr} AA^* / n . \tag{5}$$

For normal A equality holds if and only if $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n|$, which is so if and only if $A = cU$, for some unitary U and scalar c .

Proof: Inequality (5) follows immediately from (9) below and the equality conditions are clear from Theorem 11:0 (6).

□

The following result follows from (9), below:

(c) For $|\lambda_k|$:

Given $\operatorname{tr} AA^*$,

$$|\lambda_k|^2 \leq \operatorname{tr} AA^* / k \quad , \quad k = 1,2,\cdots,n \quad . \tag{6}$$

□

(d) For the spread:

Given $\operatorname{tr} AA^*$,

$$sp(A) \leq (2 \operatorname{tr} AA^*)^{1/2} \quad . \tag{7}$$

Proof: We have $sp(A) \leq |\lambda_1| + |\lambda_2| \leq \sigma_1 + \sigma_2$. Also, from the Cauchy-Schwarz inequality $(\sigma_1+\sigma_2)^2 \leq 2(\sigma_1{}^2+\sigma_2{}^2) \leq 2 \operatorname{tr} AA^*$, which completes the proof. Inequality (7) also follows from 1(6) below.

□

(e) For sum of eigenvalues:

Given $\operatorname{tr} AA^*$,

$$\sum_i |\lambda_i| \leq (n \operatorname{tr} AA^*)^{1/2} \quad , \tag{8}$$

$$\sum_i |\lambda_i|^2 \leq \operatorname{tr} AA^* \quad . \tag{9}$$

Equality holds in (8) if and only if A is a scalar matrix. Equality holds in (9) if and only if A is normal.

Proof: Inequality (9) is immediate from (3) and Theorem II:4:1 (1). Also from Theorem II:4:1 (2) $\sigma_i = |\lambda_i|$, $i = 1,2,\cdots,n$, if and only if A is normal. Further, from the Cauchy-Schwarz inequality we have, $|\sum_i \lambda_i|^2 \leq n \sum|\lambda_i|^2$ and equality holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. Now (8) follows from (9), while the equality condition follows

from Theorem 11:0 (7).

□

(f) For product of eigenvalues:

Given $\operatorname{tr} AA^*$,

$$|\lambda_1, \lambda_2, \cdots, \lambda_k| \leq \left(\frac{(\operatorname{tr} AA^*)^k}{k!}\right)^{1/2} \quad , \quad k = 1,2,\cdots,n \quad . \tag{10}$$

<u>Proof</u>: The proof is immediate from (6).

□

(g) For singular values:

The results given below are immediate consequences of (3).

Given $\operatorname{tr} AA^*$,

$$\operatorname{tr} AA^* / n \leq \sigma_1{}^2 \leq \operatorname{tr} AA^* \quad ; \tag{11}$$

$$\sigma_k{}^2 \leq \operatorname{tr} AA^* / k \quad ; \quad k = 1,2,\cdots,n \quad ; \tag{12}$$

$$\sum_i \sigma_i{}^2 = \operatorname{tr} AA^* \quad . \tag{13}$$

□

§4:1

In this section we shall give results which involve $\operatorname{tr} AA^*$, $\operatorname{tr} A$ and n. Most of the results are proved in [63]. Their proofs involve the inequalities,

$$\sum_i |\lambda_i|^2 \leq \operatorname{tr} AA^* \quad , \quad |\operatorname{tr} A| \leq \sum_i |\lambda_i| \leq (n \operatorname{tr} AA^*)^{1/2} \quad ,$$

and the Cauchy-Schwarz inequality. We shall omit these proofs. Following [63] we define:

$$m = \operatorname{tr} A / n \quad \text{and} \quad s_A^{\,2} = \operatorname{tr} AA^* / n - |m|^2. \tag{1}$$

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

Given $\operatorname{tr} AA^*$ and $\operatorname{tr} A$,

$$|m| \leq |\lambda_1| \leq |m| + s_A (n-1)^{1/2} \quad . \tag{2}$$

Equality holds on the right if and only if A is normal, $\lambda_2 = \lambda_3 = \cdots = \lambda_n$ and $\lambda_1 = cm$ for some $c \geq 1$.

<u>Proof</u>: See [63].

□

(b) For $|\lambda_n| = \min_i |\lambda_i|$:

Given $\operatorname{tr} AA^*$ and $\operatorname{tr} A$,

$$|m| - s_A (n-1)^{1/2} \leq |\lambda_n| \leq \left(\frac{\operatorname{tr} AA^*}{n}\right)^{1/2} \quad . \tag{3}$$

Equality holds on the left if and only if A is normal, $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$ and $\lambda_1 = cm$ for some real nonnegative scalar $c \leq 1$. Equality holds on the right if and only if A is normal and $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n|$.

Proof: See [63].

□

(c) For $|\lambda_k|$:

Given $1 \leq k \leq n$,

$$|m| - s_A\left(\frac{k-1}{n-k+1}\right)^{1/2} \leq |\lambda_k| \leq \left(\frac{\operatorname{tr} AA^*}{n}\right)^{1/2} + s_A\left(\frac{n-k}{k}\right)^{1/2} \quad . \tag{4}$$

Equality holds on the left if and only if A is normal and

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} \quad \text{and} \quad \lambda_k = \lambda_{k+1} = \cdots = \lambda_n = c\lambda_1 \quad , \tag{5}$$

for some nonnegative c . Equality holds on the right if and only if A is a scalar matrix.

Proof: See [63].

□

(d) For the spread:

For any A ,

$$sp(A) \leq (2 \operatorname{tr} AA^* - \frac{2}{n} |\operatorname{tr} A|^2)^{1/2} \quad . \tag{6}$$

Equality holds if and only if A is normal and $(n-2)$ eigenvalues are equal to each other and the arithmetic mean of the remaining two.

<u>Proof</u>: See [40].

□

(e) For sum of eigenvalues:

Defining,

$$|\lambda|_{(k,\ell)} = \frac{1}{\ell-k+1} \sum_{j=k}^{\ell} |\lambda_j| \quad , \tag{7}$$

m and s_A as in (1),

$$|m| - s_A\left(\frac{k-1}{n-k+1}\right)^{1/2} \le |\lambda|_{(k,\ell)} \le \left(\frac{\operatorname{tr} AA^*}{n}\right)^{1/2} + s_A\left(\frac{n-\ell}{\ell}\right)^{1/2} . \tag{8}$$

Equality holds on the left if and only if A is normal,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} \quad \text{and} \quad \lambda_k = \lambda_{k+1} = \cdots = \lambda_n = c\lambda_1 \quad , \tag{9}$$

with c real and nonnegative. Equality holds on the right if and only if A is a scalar matrix.

<u>Proof</u>: See [63].

□

§4:2

Given the diagonal elements, we know the $\operatorname{tr} A$. Hence, all the results of Section 1 hold. In addition, we have the following:

(g) For product of eigenvalues:

If $a_{ii} = 1$, $i = 1,2,\cdots,n$ and $t = (\operatorname{tr} AA^* - n) < 1$ then:

$$\exp(t^{1/2})\ (1-t^{1/2}) \leq |\det A| \quad . \tag{1}$$

<u>Proof</u>: See [7, pg. 72]. □

§4:3

When $\operatorname{tr} AA^*$ and $\operatorname{tr} A^2$ are known, clearly the results of Section 0 and 3:0 hold. In addition, the following bounds for the sum of the squares of real and imaginary parts of the eigenvalues of A are given in [39, pg. 309]:

(e) For sum of eigenvalues:

With $B = \frac{1}{2}(A + A^*)$ and $C = \frac{1}{2i}(A - A^*)$,

$$\sum_i (\mathrm{Re}(\lambda_i))^2 \leq \operatorname{tr} B^2 = \frac{1}{2}(\operatorname{tr} AA^* + \mathrm{Re}(\operatorname{tr} A^2)) \quad ; \tag{1}$$

$$\sum_i (\mathrm{Im}(\lambda_k))^2 \leq \operatorname{tr} C^2 = \frac{1}{2}(\operatorname{tr} AA^* - \mathrm{Re}(\operatorname{tr} A^2)) \quad , \tag{2}$$

with equality in (1) if and only if equality holds in (2) if and only if A is normal.

<u>Proof</u>: By Schur's triangularization theorem, there exists a unitary matrix U and an upper triangular matrix T, with eigenvalues of A along the diagonal such that $A = UTU^*$. Thus, $A \pm A^* = U(T \pm T^*)U^*$ and $\operatorname{tr}(A + A^*)^2 = \operatorname{tr}(U(T+T^*)^2U^*) = \operatorname{tr}(T + T^*)^2 = \operatorname{tr} T^2 + \operatorname{tr} T^{*2} + 2 \operatorname{tr} TT^*$. Thus,

$$\operatorname{tr} B^2 \geq \frac{1}{4}(\operatorname{tr} T^2 + \operatorname{tr} T^{*2}) = \frac{1}{2} \sum_i (\mathrm{Re}(\lambda_i))^2 \quad . \tag{3}$$

Now, (1) follows by observing that $\operatorname{tr} B^2 = \frac{1}{2}(\operatorname{tr} AA^* + \mathrm{Re}(\operatorname{tr} A^2))$ and that equality holds in (3) if and only if A is normal.

Inequality (2) follows by repeating the above reasoning for $\frac{1}{2i}(A - A^*)$, instead of $\frac{1}{2}(A + A^*)$.

□

§4:3.1

In this section we shall give bounds which involve $\operatorname{tr} AA^*$, $\operatorname{tr} A^2$ and $\operatorname{tr} A$. All the results given below, are proved in [63]. We will need the following notation:

Let the moduli, real and imaginary parts of the eigenvalues of A, arranged in decreasing order be denoted by $\lambda_i^{(A)}$, $\lambda_j^{(B)}$ and $\lambda_k^{(C)}$, respectively. That is,

$$\lambda_1^{(A)} \geq \lambda_2^{(A)} \geq \cdots \geq \lambda_n^{(A)} ; \tag{1}$$

$$\lambda_1^{(B)} \geq \lambda_2^{(B)} \geq \cdots \geq \lambda_n^{(B)} ; \text{ and} \tag{2}$$

$$\lambda_1^{(C)} \geq \lambda_2^{(C)} \geq \cdots \geq \lambda_n^{(C)} , \tag{3}$$

where,

$$\lambda_i^{(A)} = |\lambda_i| , \lambda_i^{(B)} = \operatorname{Re}(\lambda_i) , \text{ and } \lambda_i^{(C)} = \operatorname{Im}(\lambda_i) . \tag{4}$$

Also we recall that,

$\operatorname{tr} B^2 = \frac{1}{2}(\operatorname{tr} AA^* + \operatorname{Re}(\operatorname{tr} A^2))$ and $\operatorname{tr} C^2 = \frac{1}{2}(\operatorname{tr} AA^* - \operatorname{Re}(\operatorname{tr} A^2))$.

(c) For λ_k :

(i) Let A be an nxn complex matrix. Define

$m_B = \operatorname{Re}(\operatorname{tr} A) / n = \operatorname{tr} B / n$, $m_C = \operatorname{Im}(\operatorname{tr} A) / n = \operatorname{tr} C / n$,

$$s_B^{\;2} = \mathrm{tr}\; B^2 / n - m_B^{\;2} \;,\; s_C^{\;2} = \mathrm{tr}\; C^2 / n - m_C^{\;2} \;. \tag{5}$$

Then, for $T = B$ or C ,

$$\lambda_1^{(T)} \leq m_T + s_T\,(n-1)^{1/2} \;. \tag{6}$$

Equality holds if and only if A is normal and

$$\lambda_2^{(T)} = \lambda_3^{(T)} = \cdots = \lambda_n^{(T)} \;. \tag{7}$$

Proof: See [63].

□

(ii) With m_T , s_T as above, for $T = B$ or C ,

$$m_T - s_T(n-1)^{1/2} \leq \lambda_n^{(T)} \;. \tag{8}$$

Equality holds if and only if A is normal and

$$\lambda_1^{(T)} = \lambda_2^{(T)} = \cdots = \lambda_{n-1}^{(T)} \;. \tag{9}$$

Proof: See [63].

□

(iii) With m_T , s_T as in (5), for $T = B$ or C and $1 \leq k \leq n$,

$$m_T - s_T\left(\frac{k-1}{n-k+1}\right)^{1/2} \leq \lambda_k^{(T)} \leq m_T + s_T\left(\frac{n-k}{k}\right)^{1/2} \;. \tag{10}$$

Equality holds on the left if and only if A is normal and

$$\lambda_1^{(T)} = \lambda_2^{(T)} = \cdots = \lambda_{k-1}^{(T)} \text{ and } \lambda_k^{(T)} = \lambda_{k+1}^{(T)} = \cdots = \lambda_n^{(T)}. \quad (11)$$

Equality holds on the right if and only if A is normal and

$$\lambda_1^{(T)} = \lambda_2^{(T)} = \cdots = \lambda_k^{(T)} \text{ and } \lambda_{k+1}^{(T)} = \lambda_{k+2}^{(T)} = \cdots = \lambda_n^{(T)}. \quad (12)$$

Proof: See [63].

□

(e) For sum of eigenvalues:

(i) With m_T , s_T for $T = B,C$, given by (5) and

$$\lambda_{(k,\ell)}^{(B)} = \frac{1}{\ell-k+1} \sum_{j=k}^{\ell} \lambda_j^{(B)} \quad ; \qquad (13)$$

$$\lambda_{(k,\ell)}^{(C)} = \frac{1}{\ell-k+1} \sum_{j=k}^{\ell} \lambda_j^{(C)} \quad , \qquad (14)$$

we have:

$$m_T - s_T\left(\frac{k-1}{n-k+1}\right)^{1/2} \leq \lambda_{(k,\ell)}^{(T)} \leq m_T + s_T\left(\frac{n-\ell}{\ell}\right)^{1/2} .$$

When $(k,\ell) = (1,n)$ the inequality string collapses. When $(k,\ell) \neq (1,n)$, equality holds on the left if and only if A is normal and (11) holds. Further, equality holds on the right if and only if A is normal and,

$$\lambda_1^{(T)} = \lambda_2^{(T)} = \cdots = \lambda_\ell^{(T)} \text{ and } \lambda_{\ell+1}^{(T)} = \lambda_{\ell+2}^{(T)} = \cdots = \lambda_n^{(T)} . \quad (15)$$

Proof: See [63].

□

(ii) If $1 \leq k < \ell \leq n$, then for $T = A,B$ or C ,

$$\lambda_k^{(T)} - \lambda_\ell^{(T)} \leq s_T \, n^{1/2} \left(\frac{1}{k} + \frac{1}{n-\ell+1}\right)^{1/2} . \tag{16}$$

Equality holds if and only if A is normal and

$$\begin{aligned} \lambda_1^{(T)} &= \lambda_2^{(T)} = \cdots = \lambda_k^{(T)} \; ; \\ \lambda_{k+1}^{(T)} &= \lambda_{k+2}^{(T)} = \cdots = \lambda_{\ell-1}^{(T)} = m_T \; ; \\ \lambda_\ell^{(T)} &= \lambda_{\ell+1}^{(T)} = \cdots = \lambda_n^{(T)} \; , \end{aligned} \tag{17}$$

where $m_A = \dfrac{\sum_1^n |\lambda_j|}{n}$ and for $T = A$,

$$\lambda_j = \alpha_j > 1 \quad , \quad j = 2,\cdots,n \quad , \tag{18}$$

for some nonnegative scalars α_j , $j = 1,2,\cdots,n$.

<u>Proof</u>: See [63].

□

CHAPTER 5

TRACE $(AA^* - A^*A)^2$

§5:0 Preliminaries.

We recall that the Euclidean (Frobenius) norm of a matrix A, is given by $||A||^2 = \text{tr } AA^*$. Let

$$D = AA^* - A^*A \quad . \tag{1}$$

Clearly D is Hermitian and therefore,

$$||D||^2 = \text{tr } D^2 = \text{tr}(AA^* - A^*A)^2 \quad .$$

By definition, A is normal if and only if $D = 0$. Further, since AA^* and A^*A have the same diagonal elements, $\text{tr } D = 0$. Thus, for a nonnormal matrix A, D has both positive as well as negative eigenvalues. Finally, it is well-known that D is positive semidefinite if and only if A is normal (e.g. see [16]). For if D is positive semidefinite, then $\text{tr } D = 0$ implies that all the eigenvalues of D are zero. Therefore, from $0 = \text{tr } D^2 = \sum_{i,j} |d_{ij}|^2$, we have $d_{ij} = 0$, $i,j = 1,2,\cdots,n$. Thus A is normal. The converse is trivial.

Now, we give some bounds involving $||AA^* - A^*A||$, and n.

(g) For singular values:

Given A,

$$||AA^* - A^*A|| \leq 2^{1/2}\, n\, \sigma_1^2 \quad , \tag{2}$$

$$||AA^* - A^*A|| \leq 2^{1/2} \sum_i \sigma_i^2 \quad . \tag{3}$$

<u>Proof</u>: Inequality (2) is immediate from (3). To prove (3), we observe that,

$$\begin{aligned} ||AA^* - A^*A||^2 &= tr(AA^* - A^*A)^2 = 2\ tr(AA^*)^2 - 2\ tr(A^2A^{*2}) \\ &= 2\ ||AA^*||^2 - 2||A^2||^2 \\ &\leq 2\ ||AA^*||^2 \leq 2\ ||A||^4 \quad , \end{aligned}$$

which completes the proof of (3).

□

§5:4

In this section we shall give bounds for the sum of the eigenvalues, which involve, $||A||$ and $||AA^* - A^*A||$.

(e) For sum of eigenvalues:

(i) With D as in (1),

$$\sum_i |\lambda_i|^2 \leq (||A||^4 - \frac{1}{2} ||D||^2)^{1/2} \quad . \tag{1}$$

Equality holds if and only if

$$A = \alpha(vw^* + r\, wv^*) \quad , \tag{2}$$

where $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, $0 \leq r < 1$, and u,w are orthonormal vectors.

<u>Proof</u>: See [25]. □

(ii) Given $||A||$ and $||D||$,

$$||A||^2 - (\frac{n^3-n}{12})^{1/2} ||D|| \leq \sum_i |\lambda_i|^2 \quad . \tag{3}$$

<u>Proof</u>: See [20]. □

§5:4.1

In this section we shall give bounds which involve $||A||$, $||AA^* - A^*A||$ and $\operatorname{tr} A$. All of the results to follow are proved in [64]. We shall need the following notation:

Suppose $\lambda_1, \lambda_2, \cdots, \lambda_n$ are complex eigenvalues of A. Define,

$$\lambda_j^A = |\lambda_i| \quad , \quad \lambda_k^B = \operatorname{Re}(\lambda_i) \quad , \quad \lambda_\ell^C = \operatorname{Im}(\lambda_i) \quad , \tag{1}$$

so that the ordered vectors (λ_j^T) satisfy

$$\lambda_1^T \geq \lambda_2^T \geq \cdots \geq \lambda_n^T \quad , \quad T = A,B,C \quad ,$$

and

$$\sum_i |\lambda_i^A|^2 = \sum_i |\lambda_i|^2 \ , \ \sum_i |\lambda_i^B|^2 = \sum_i (\operatorname{Re}(\lambda_i))^2 \ , \ \sum_i |\lambda_i^C|^2 = \sum_i (\operatorname{Im}(\lambda_i))^2 .$$

Further , let

$$\begin{aligned} K_A^u &= (||A||^4 - \tfrac{1}{2}\, ||D||^2)^{1/2} \quad , \\ K_A^\ell &= ||A||^2 - (\tfrac{n^3-n}{12})^{1/2} ||D|| \ , \end{aligned} \tag{2}$$

$$\begin{aligned} K_B^u &= \begin{cases} (||B||^4 - \tfrac{1}{8}\, ||D||^2)^{1/2} & \text{if } ||B|| \geq ||C|| \ , \\ ||B||^2 - \dfrac{1}{12} \dfrac{||D||^2}{||A||^2} & \text{otherwise,} \end{cases} \\ K_B^\ell &= ||B||^2 - (\tfrac{n^3-n}{48})^{1/2} \, ||D|| \ , \end{aligned} \tag{3}$$

$$K_C^u = \begin{cases} (||C||^4 - \frac{1}{8}||D||^2)^{1/2} & \text{if } ||C|| \geq ||B|| \\ ||C||^2 - \frac{1}{12}\frac{||D||^2}{||A||^2} & \text{otherwise,} \end{cases}$$

$$K_C^\ell = ||C||^2 - (\frac{n^3-n}{48})^{1/2}\,||D|| \quad . \tag{4}$$

Finally define,

$$S_T^2 = \frac{\sum_i |\lambda_i^T|^2}{n} - \frac{|\sum_i \lambda_i^T|^2}{n^2} \quad , \quad T = A,B,C \quad , \tag{5}$$

$$m_A^u = \left(\frac{K_A^u}{n}\right)^{1/2} \quad , \quad m_A^\ell = \frac{|\mathrm{tr}\ A|}{n} \quad ,$$

$$m_B^u = m_B^\ell = \frac{\mathrm{tr}\ B}{n} \quad , \quad m_C^u = m_C^\ell = \frac{\mathrm{tr}\ C}{n} \tag{6}$$

$$(S_T^u)^2 = \frac{K_T^u - |\mathrm{tr}\ T|^2/n}{n} \quad , \quad T = A,B,C$$

$$(S_T^\ell)^2 = \max\{0 \,,\, \frac{K_T^\ell - |\mathrm{tr}\ T|^2/n}{n}\} \,, T = A,B,C \quad . \tag{7}$$

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

With the above notation:

$$m_A^\ell + (n-1)^{-1/2}\, s_A^\ell \leq |\lambda_1| \leq m_A^u + (n-1)^{1/2}\, s_A^u \quad . \tag{8}$$

<u>Proof</u>: See [64].

□

(b) For $|\lambda_n| = \min_i |\lambda_i|$:

With the above notation:

$$m_A^{\ell} - (n-1)^{1/2} s_A^{u} \leq |\lambda_n| \leq m_A^{u} - (n-1)^{-1/2} s_A^{\ell} \quad . \tag{9}$$

<u>Proof</u>: See [64].

□

(c) For $|\lambda_k|$:

Given $1 \leq k \leq n$,

$$m_A^{\ell} - (\frac{k-1}{n-k+1})^{1/2} s_A^{u} \leq |\lambda_k| \leq m_A^{u} + (\frac{n-k}{k})^{1/2} s_A^{u} \quad . \tag{10}$$

<u>Proof</u>: See [64].

□

(d) For the spread:

With s_A^{ℓ} , s_A^{u} as in (7),

$$2 s_A^{\ell} \leq sp(A) \leq (2n)^{1/2} s_A^{u} \quad . \tag{11}$$

Further, if n is odd, then:

$$\frac{2n}{(n^2-1)^{1/2}} s_A^{\ell} \leq sp(A) \ .$$

<u>Proof</u>: See [64].

□

(e) For sum of eigenvalues:

For $1 \le j \le k \le n$,

$$m_A^{\ell} - \left(\frac{j-1}{n-j+1}\right)^{1/2} s_A^{u} \le \frac{1}{k-j+1} \sum_{i=j}^{k} |\lambda_i|$$

$$\le m_A^{u} + \left(\frac{n-k}{k}\right)^{1/2} s_A^{u} \quad ; \qquad (12)$$

$$m_A^{\ell} + (n-k)\ \gamma^{-1}(n-1)^{-1/2}\ s_A^{\ell} \le \frac{1}{k} \sum_{i=1}^{k} |\lambda_i| \ ,$$

where $\gamma = \max(k,n-k)$; with $\gamma = \max(n-k+1,k-1)$,

$$\frac{1}{n-k+1} \sum_{i=k}^{n} |\lambda_i| \le m_A^{u} - (k-1)\gamma^{-1}\ (n-1)^{-1/2}\ s_A^{\ell} \ ;$$

and

$$|\lambda_j| - |\lambda_k| \le \left(\frac{1}{j} + \frac{1}{n-k+1}\right)^{1/2} n^{1/2}\ s_A^{u} \ . \qquad (13)$$

Proof: See [64].

□

(f) For product of eigenvalues:

Given A ,

$$\prod_1^k |\lambda_i| \le \prod_1^k \left(m_A^{u} + \left(\frac{n-k}{k}\right)^{1/2}\right) s_A^{u} \ , \quad k = 1,2,\cdots,n \ .$$

Proof: The proof is immediate from (10).

□

(g) For singular values:

Given A and m_A^{ℓ}, m_A^{u}, s_A^{ℓ}, s_A^{u} as in (6) and (7),

$$m_A^{\ell} + (n-1)^{-1/2} s_A^{\ell} \leq \sigma_1 \quad ; \tag{14}$$

$$\sigma_n \leq m_a^{u} - (n-1)^{-1/2} s_A^{\ell} \quad . \tag{15}$$

Proof: Since $|\lambda_1| \leq \sigma_1$ and $|\lambda_n| \geq \sigma_n$, the above inequalities follow from (8) and (10).

□

(h) For the condition number:

If $m_A^{\ell} - (n-1)^{1/2} s_A^{u} > 0$, then,

$$\frac{m_A^{\ell} + s_A^{\ell} / (n-1)^{1/2}}{m_A^{\ell} - (n-1)^{1/2} s_A^{u}} \leq \frac{|\lambda_1|}{|\lambda_n|} \leq c(A) \quad . \tag{16}$$

Proof: As $|\lambda_1| \leq \sigma_1$ and $|\lambda_n| \geq \sigma_n$, the proof is immediate from (8) and (9).

□

§5:4.3

The bounds for the sum of the squares of real and imaginary parts of the eigenvalues of A, given in 4:3 can be improved when $||AA^* - A^*A||$, $||A||$ and $\operatorname{tr} A^2$ are known:

(e) For sum of eigenvalues:

With (λ_i^T), K_T^ℓ and K_T^u as given by 4.1 (1), 4.1 (2), 4.1 (3) and 4.1 (4),

$$K_T^\ell \leq \sum_i |\lambda_i^T|^2 \leq K_T^u \quad , \quad T = B, C \quad .$$

<u>Proof</u>: See [64]. □

§5:4.3.1

In this section we shall state bounds for the real and imaginary parts of the eigenvalues of A, which involve $||A||$, $||AA^* - A^*A||$, $\text{tr } A^2$ and $\text{tr } A$.

All the bounds given below, are proved in [64]. We shall omit their proofs. We recall that

$$B = \frac{1}{2}(A + A^*) \quad , \quad C = \frac{1}{2i}(A - A^*) \quad ,$$

$$\text{tr } B^2 = \frac{1}{2}(\text{tr } AA^* + \text{Re}(\text{tr } A^2)) \quad \text{and} \quad \text{tr } C^2 = \frac{1}{2}(\text{tr } AA^* - \text{Re}(\text{tr } A^2)) \; .$$

Also, m_T^{ℓ}, m_T^{u}, s_T^{ℓ}, s_T^{u}, for $T = B,C$ are given by 4.1 (5), 4.1 (6) and 4.1 (7). Lastly, we assume that λ_i^T, $T = B,C$ are given by 4.1 (1).

(a) For λ_1 :

Given A, then for $T = B,C$:

$$m_T^{\ell} + (n-1)^{-1/2} s_T^{\ell} \leq \lambda_1^T \leq m_T^{u} + (n-1)^{1/2} s_T^{u} \; . \qquad (1)$$

<u>Proof</u>: See [64].

(b) For λ_n :

Given A, for $T = B,C$:

$$m_T^{\ell} - (n-1)^{1/2} s_T^{u} \leq \lambda_n^T \leq m_T^{u} - (n-1)^{-1/2} s_T^{\ell} \; . \qquad (2)$$

<u>Proof</u>: See [64].

□

(c) For λ_k :

For $T = B,C$,

$$m_T^{\ell} - \left(\frac{k-1}{n-k+1}\right)^{1/2} s_T^{u} \leq \lambda_k^{T} \leq m_T^{u} + \left(\frac{n-k}{k}\right)^{1/2} s_T^{u} \quad . \tag{3}$$

<u>Proof</u>: See [64].

□

(d) For the spread:

Given A , then:

$$\begin{aligned} 2\, s_B^{\ell} &\leq sp_R(A) \leq (2n)^{1/2}\, s_B^{u} \quad ; \\ 2\, s_C^{\ell} &\leq sp_I(A) \leq (2n)^{1/2}\, s_C^{u} \quad . \end{aligned} \tag{4}$$

Further, if n is odd:

$$2\, s_B^{\ell}\, n\, /\, (n^2-1)^{1/2} \leq sp_R(A) \quad ,$$

$$2\, s_C^{\ell}\, n\, /\, (n^2-1)^{1/2} \leq sp_I(A) \quad .$$

<u>Proof</u>: See [64].

□

(e) For sum of eigenvalues:

Given $1 \leq i \leq j \leq k \leq n$, then for $T = B,C$,

$$m_T^{\ell} - \left(\frac{j-1}{n-j+1}\right)^{1/2} s_T^{u} \leq \frac{1}{k-j+1} \sum_{i=j}^{k} \lambda_i^T \leq m_T^{u} + \left(\frac{n-k}{k}\right)^{1/2} s_T^{u} \quad ;$$

$$m_T^{\ell} + (n-k)\, \gamma^{-1} (n-1)^{-1/2} s_T^{\ell} \leq \frac{1}{k} \sum_{i=1}^{k} \lambda_i^T \quad ,$$

where $\gamma = \max(k,n-k)$;

$$\frac{1}{n-k+1} \sum_{i=k}^{n} \lambda_i^T \leq m_T^{u} - (k-1)\, \gamma^{-1} (n-1)^{-1/2} s_T^{\ell} \quad ,$$

where $\gamma = \max(n-k+1,k-1)$; and

$$|\lambda_j^T - \lambda_k^T| \leq \left(\frac{1}{j} + \frac{1}{n-k+1}\right)^{1/2} n^{1/2} s_T^{u} \quad .$$

Proof: See [64].

□

(d) For singular values:

For $T = B,C$,

$$m_T^{\ell} + (n-1)^{-1/2} s_T^{\ell} \leq \sigma_1 \quad .$$

Proof: The inequality is immediate from 4.3.1 (1), as $\mathrm{Re}(\lambda_1) \leq |\lambda_1| \leq \sigma_1$.

□

CHAPTER 6

ROW AND COLUMN SUMS

§6:0 Preliminaries.

Given a matrix A, its row and column sums can be calculated quite easily. Thus, the bounds for eigenvalues involving row or column sums can be quite useful. Perhaps, this is the reason that many inequalities relating eigenvalues and row (column) sums are known. The row (column) sums together with the diagonal elements yield very useful results. The Gerschgorin Theorem (see Section 2) is one example.

We shall use the following standard notation for the various sums:

$$R_i = \sum_j |a_{ij}| \quad , \quad C_j = \sum_i |a_{ij}| \quad , \quad 1 \le i \, , \, j \le n \quad , \tag{1}$$

$$P_i = R_i - |a_{ii}| \, , \quad Q_j = C_j - |a_{jj}| \, ; \quad 1 \le i \, , \, j \le n \quad , \tag{2}$$

$$R = \max_i R_i \quad \text{and} \quad C = \max_j C_j \quad . \tag{3}$$

Also,

$$r_i = \sum_j \frac{|a_{ij}| \, x_j}{x_i} \quad , \quad c_j = \sum_i \frac{|a_{ij}| \, x_i}{x_j} \quad , \tag{4}$$

where, $x_i > 0$ and $i,j = 1,2,\cdots,n$. Further, we shall assume that $i_1,i_2,\cdots,i_n$ is a permutation of the integers $1,2,\cdots,n$ such that,

$$r_{i_1} \ge r_{i_2} \ge \cdots \ge r_{i_n} \quad . \tag{5}$$

As always we shall assume that the complex eigenvalues of A are ordered as,

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \quad . \tag{6}$$

Before we give bounds for eigenvalues, observe that the eigenvalues of A and A' are the same. Thus, bounds involving row sums are also true for column sums.

Now, we give the bounds which involve only row and column sums.

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

With R as in (3),

$$|\lambda_1| \leq R \quad . \tag{7}$$

If A is irreducible, $|\lambda_1| = R$ if and only if $R_1 = R_2 = \cdots = R_n$ and $A = e^{i\theta} D^{-1} PD$, where D is a diagonal matrix with $|d_{ii}| = 1$, $P = (|a_{ij}|)$ and θ is a real number. In fact if r_i is as in (4), then:

$$|\lambda_1| \leq \max_i |r_i| \ . \tag{8}$$

<u>Proof</u>: Inequality (7) follows from Gerschgorin's Theorem below. The conditions for equality are given in [50]. Inequality (8) follows when (7) is applied to $X^{-1}AX$, $X = \mathrm{diag}(x_1, x_2, \cdots, x_n)$.

□

(b) For $|\lambda_n| = \min_i |\lambda_i|$:

If R_i or C_i is zero for some $1 \leq i \leq n$, then

$$|\lambda_n| = 0 \quad . \tag{9}$$

Proof: The proof follows from (15) below, or more simply that A must be singular.

□

(d) For the spread:

With r_{i_k} as in (5),

$$sp(A) \leq |\lambda_1| + |\lambda_2| \leq r_{i_1} + r_{i_2} \quad . \tag{10}$$

Proof: Since $sp(A) \leq |\lambda_1| + |\lambda_2|$, (9) follows from (11) below, when $\alpha = 1$ and $\ell = 2$.

□

(e) For sum of eigenvalues:

(i) With r_{i_k} , $k = 1,2,\cdots,n$ as in (5),

$$\sum_{k=1}^{\ell} |\lambda_k|^{\alpha} \leq \sum_{k=1}^{\ell} r_{i_k}^{\alpha} \quad , \quad \alpha \geq 0 \ , \ 1 \leq \ell \leq n \quad . \tag{11}$$

Proof: See [50].

□

(ii) Given A,

$$\sum_i |\lambda_i|^2 \le \sum_i R_i^2 - (p-q)^2 \quad , \tag{12}$$

where,

$$p = \frac{1}{n} \Big(\sum_{i<k} |c_i - c_k|^2 \Big)^{1/2} \quad , \quad q = \frac{1}{n} \Big(\sum_{i<k} |\gamma_i - \gamma_k|^2 \Big)^{1/2} \quad ,$$

$$c_i = \sum_j a_{ji} \quad \text{and} \quad r_i = \sum_j a_{ij} \quad .$$

<u>Proof</u>: From (21) below, $\sum_i \sigma_i^2 = ||A||^2 \le \sum_i R_i^2$. Now, the result follows from 4 (1), below.

□

(f) For product of eigenvalues:

(i) Given $1 \le \ell \le n$ and r_{i_k} as in (5),

$$\prod_1^{\ell} |\lambda_k| \le \prod_1^{\ell} r_{i_k} \quad . \tag{13}$$

In particular,

$$\prod_1^{\ell} |\lambda_k| \le \prod_1^{\ell} R_{i_k} \quad , \tag{14}$$

and for $\ell = n$,

$$|\det A| \le \prod_1^{n} R_i \quad . \tag{15}$$

When A is irreducible,

$$\prod_1^{k} |\lambda_i| = \prod_1^{k} R_i \quad \text{for} \quad k = 1,2,\cdots,\ell \le n \quad ,$$

if and only if

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_\ell| = R_1 = R_2 = \cdots = R_n \quad .$$

Proof: See [50]. □

(ii) If R_i or $C_i = 0$ for some $1 \leq i \leq n$, then:

$$|\det A| = 0 \quad .$$

Proof: Proof is immediate from (15). □

(g) For singular values:

(i) Given A ,

$$R / n^{1/2} \leq \max_i \left(\sum_j |a_{ij}|^2\right)^{1/2} \leq \sigma_1 \leq \frac{1}{2}(t_1 + t_2) \quad ; \tag{16}$$

$$\sigma_n \leq \min_i \left(\sum_j |a_{ij}|^2\right)^{1/2} \leq \min_i R_i \quad , \tag{17}$$

$$\sum_1^k \sigma_i^{\alpha} \leq \frac{1}{2} \sum_1^{2k} t_i^{\alpha} \quad , \quad \alpha \geq 0 \ , \ k = 1,2,\cdots,n \quad ; \tag{18}$$

and

$$\left(\sum_i R_i^2 / n\right)^{1/2} \leq \sum_i \sigma_i \leq \sum_i R_i \quad , \tag{19}$$

where, t_i , $i = 1,2,\cdots,2n$ are R_i , C_i , $i = 1,2,\cdots,n$ arranged in decreasing order.

Proof: An application of the Cauchy-Schwarz inequality yields $R_i / n^{1/2} \le (\sum_i |a_{ij}|^2)^{1/2}$. The second inequality in (16) and the inequality on the left in (17) follows from Theorems II:4:4 (1) and II:4:4 (6). Further, it is known that the eigenvalues of the matrix,

$$P = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

are $\pm\sigma_1, \pm\sigma_2, \cdots, \pm\sigma_n$, for if $AA^*x = \sigma^2 x$, then setting $y = \pm\frac{1}{\sigma}A^*x$, we have $Ay = \pm\sigma x$. Thus with $z' = (x,y)'$, we have

$$Pz = (Ay, A^*x)' = (\pm\sigma x, \pm\sigma y)' = \pm\sigma z \ .$$

Now, the inequality on the right in (16) and inequality (18) follow from (11) with $x_i = 1$, $1 \le i \le n$. Also, with $\alpha = 1$ the inequality on the right in (19) follows from (18) as $\sum_i R_i = \sum_i C_i$. The inequality on the left follows from (20) below.

□

(ii) Given A ,

$$(\sum_i R_i / n)^2 \le \sum_i R_i^2 / n \le \sum_i \sigma_i^2 = \text{tr } AA^* \le \sum_i R_i^2 \ . \tag{20}$$

Proof: Since $\sum_i \sigma_i^2 = \text{tr } AA^* = \sum_{i,j} |a_{ij}|^2$, we have $\sum_i \sigma_i^2 = \sum_i \sum_j |a_{ij}|^2 \le \sum_i R_i^2$. Also, $\sum_i R_i^2 = \sum_i (\sum_j |a_{ij}|)^2 \le n \sum_{i,j} |a_{ij}|^2 = n \sum_i \sigma_i^2$, by the Cauchy-Schwarz inequality.

Another application of the Cauchy-Schwarz inequality yields
$(\sum_i R_i / n)^2 \leq \sum_i R_i^2 / n$.

□

(iii) Given A ,

$$\prod_i \sigma_i \leq \prod_i R_i \quad . \tag{21}$$

<u>Proof</u>: From Theorem II:4:1 (1), $\prod_i \sigma_i = |\det A|$. Now (21) follows from (15).

□

§6:2

In this section we shall give bounds for eigenvalues when the diagonal elements and row (column) sums are known. In particular, we discuss the diagonally dominant matrices:

<u>Definition 1</u>: Given an $n \times n$ matrix A, if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| = P_i \quad , \quad i = 1,2,\cdots,n \quad , \tag{1}$$

then we say A is <u>weakly diagonally dominant</u>. If, all the inequalities in (1) are strict, A is called <u>diagonally dominant</u>.

□

Clearly, one may also define diagonally dominant matrices which involve Q_i's instead of P_i's, $i = 1,2,\cdots,n$.

Now, we give the bounds for the eigenvalues. We note that all the results involving R_i (or P_i) hold for C_i (or Q_i) also.

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

If $0 \leq \alpha \leq 1$, then:

$$|\lambda_1| \leq \max_i \left(|a_{ii}| + P_i^{\alpha} Q_i^{1-\alpha}\right) \quad ; \tag{2}$$

$$|\lambda_1| \leq \max_i \left(|a_{ii}| + \alpha P_i + (1-\alpha) Q_i\right) \quad , \tag{3}$$

where P_i and Q_i are given by 0.(2).

<u>Proof</u>: See [31, pg. 151].

□

(b) For $|\lambda_n| = \min_i |\lambda_i|$:

If A is diagonally dominant, then:

$$0 < \min_i(|a_{ii}| - P_i) \leq |\lambda_n| \quad . \tag{4}$$

<u>Proof</u>: Inequality (4) is immediate from the Gerschgorin's Theorem below.

□

The following result is given in [31, pg. 159]:

(c) For λ_i :

(i) Let A be such that

$$\mathrm{Re}(a_{ii}) \leq -P_i \quad , \quad i = 1,2,\cdots,n \quad . \tag{5}$$

Then,

$$\mathrm{Re}(\lambda_i) \leq 0 \quad , \quad i = 1,2,\cdots,n \quad . \tag{6}$$

Further, if the inequalities (5) are strict, then so are the inequalities (6).

<u>Proof</u>: If λ is any eigenvalue of A then from the Gerschgorin's Theorem below, there exists $1 \leq i \leq n$ such that,

$$\mathrm{Re}(\lambda) - \mathrm{Re}(a_{ii}) \leq P_i \quad .$$

Now, the proof follows from the inequalities (5).

□

(ii) (Gerschgorin's Theorem). Let P_i be as in 0 (2). Then every eigenvalue of A lies in at least one of the disks:

$$|z - a_{ii}| \leq P_i \quad , \quad i = 1,2,\cdots,n \quad , \tag{7}$$

in the complex plane. Furthermore, a set of m disks having no point in common with the remaining $n-m$ disks contains exactly m eigenvalues of A .

Proof: If λ_k is an eigenvalue of A , then $\det(A-\lambda_k I) = 0$ Now (7) follows from the Lévy-Desplanques Theorem, below, since $A - \lambda_k I$ cannot be diagonally dominant. Further, in case a set of m disks $(m \leq n)$ has no point in common with the remaining $(n-m)$ disks, it follows by the continuity argument that this set contains exactly m eigenvalues (e.g. see [26, pg. 226]).

□

The following two results are proved in [31, pgs. 149 - 151]:

(iii) With P_i , $i = 1,2,\cdots,n$, as in 0 (2), define the $\frac{n(n-1)}{2}$ ovals of Cassini,

$$\{z \,|\, |z-a_{ii}|\,|z - a_{jj}) \leq P_i P_j\} \quad , \quad i,j = 1,2,\cdots,n \ , \ i \neq j \ . \tag{8}$$

Then, each eigenvalue of A lies in at least one of the ovals.

□

(iv) Given A , each of its eigenvalues lies on or inside one of the disks,

$$|z-a_{ii}| \leq P_i^{\alpha} Q_i^{1-\alpha} \quad , \ 0 \leq \alpha \leq 1 \ , \ i = 1,2,\cdots,n \quad . \tag{9}$$

□

(f) For product of eigenvalues:

(i) (Lévy-Desplanques Theorem). If A is diagonally dominant then $\det A \neq 0$.

Proof: If $\det A = 0$, then the system $Ax = 0$ has a nontrivial solution x . Let $|x_k| = \max_i |x_i|$. Then,

$$|a_{kk}|\,|x_k| = |\sum_{j \neq k} a_{kj}\, x_j| \leq |x_k|\, P_k$$

i.e. $|a_{kk}| \leq P_k$, which contradicts the fact that A is diagonally dominant. Hence $\det A \neq 0$.

□

The above diagonally dominant assumption can be weakened for irreducible matrices:

(ii) If A is irreducible and

$$|a_{ii}| \geq P_i \quad , \quad i = 1,2,\cdots,n \quad , \tag{10}$$

with equality in at most $(n-1)$ cases then:

$$\det A \neq 0 \quad .$$

Proof: See [54].

□

Below, we give several other conditions which guarantee the nonsingularity of A .

The following results are proved in [21, pgs. 149 - 150]:

(iii) If A is such that

$$|a_{ii}|\,|a_{jj}| > P_i P_j \quad , \quad i,j = 1,2,\cdots,n \;,\; i \neq j \quad , \tag{11}$$

then $\det A \neq 0$.

□

(iv) Given $0 \leq \alpha \leq 1$, if

$$|a_{ii}| > P_i^{\alpha}\, Q_i^{1-\alpha} \quad , \quad i = 1,2,\cdots,n \quad , \tag{12}$$

then $\det A \neq 0$.

□

The following result is cited in [2]:

(v) Let $R_1 = P_1$ and

$$R_i = \sum_{t=1}^{i-1} |a_{it}| \frac{R_t}{|a_{tt}|} + \sum_{t=i+1}^{n} |a_{it}| \quad , \quad i = 2,3,\cdots,n \quad . \tag{13}$$

If,

$$|a_{ii}| > R_i \quad , \quad i = 1,2,\cdots,n \quad ,$$

then $\det A \neq 0$.

□

(vi) If (v) above holds and for $2 \leq i \leq n$,

$$\ell_i = \sum_{t=1}^{i-1} |a_{it}| \frac{R_t}{|a_{tt}|} \quad \text{and} \quad L_i = |a_{i1} a_{11}^{-1}| \sum_{t=i+1}^{n} |a_{1t}|, \; ,$$

$t = i+1$, then

$$|a_{11}| \prod_{i=2}^{n} (|a_{ii}|-\ell_i+L_i) \le |\det A|$$

$$\le |a_{11}| \prod_{i=2}^{n} (|a_{ii}|+\ell_i-L_i) \quad . \qquad (14)$$

Proof: See [2].

□

(vii) If $a_{ii} = 1$, $P_i < 1$, $p = \max P_i < 1$ and $P = \sum_i P_i$ then:

$$e^P(1-p)^{P/p} \le |\det A| \quad . \qquad (15)$$

Proof: See [7, pg. 71].

□

(viii) If A is diagonally dominant then:

$$|\det A| \le \prod_j d_j \quad ,$$

where $d_j = |a_{jj}| - P_j$.

Proof: See [26, pg. 228].

□

(g) For singular values:

(i) Given A ,

$$\max(R/n^{1/2}, \max_i |a_{ii}|) \le \sigma_1 \quad . \qquad (16)$$

Proof: The proof is immediate from 0 (16) and 2:0 (3).

□

The following results are proved in [28]:

(ii) Let A be an $m\times n$ matrix. Define:

$$r_i = \sum_{j\neq i} \frac{k_j|a_{ij}|}{k_i} \; , \; c_i = \sum_{j\neq i} \frac{k_j}{k_i}|a_{ji}| \; , \; i = 1,2,\cdots,\min(m,n) \; .$$

$$s_i = \max(r_i,c_i) \quad , \quad a_i = |a_{ii}| \quad \text{and}$$

$$s = \begin{cases} \max\limits_{n+1\le i\le m} \sum\limits_{j=1}^{n} \dfrac{k_j|a_{ij}|}{k_i} \; , & \text{for } m > n \\ \max\limits_{m+1\le i\le n} \sum\limits_{j=1}^{m} \dfrac{k_j|a_{ji}|}{k_i} \; , & \text{for } m < n \; , \end{cases}$$

where $k_i > 0$, $i = 1,2,\cdots,\max(m,n)$ are any positive numbers and $a_+ = \max(0,a)$, for any real a .

Then each singular value of A lies in one of the intervals:

$$B_i = [(a_i - s_i)_+ \, , \, a_i + s_i] \quad , \quad i = 1,2,\cdots,n \quad ,$$

$$B_{n+1} = [0,s] \quad .$$

If $m = n$ or if $m > n$ and $a_i \geq s_i + s$, $i = 1,2,\cdots,n$, then B_{n+1} is not needed in the above statement. Furthermore, every interval of the union of B_i , $i = 1,2,\cdots,n+1$ (n for $m=n$) , contains exactly k singular values if it contains k intervals of $B_1,B_2,\cdots,B_n$.

□

(iii) In (ii) $B_i = 1,2,\cdots,n$ can be replaced by

$$G_i = [\ell_{i+}, u_i] \quad , \quad i = 1,2,\cdots,n \quad ,$$

where

$$\ell_i = \min\left\{\sqrt{a_i^2 - a_i r_i + \frac{c_i^2}{4}} - \frac{c_i}{2}, \sqrt{a_i^2 - a_i c_i + \frac{r_i^2}{4}} - \frac{r_i}{2}\right\} ,$$

$$u_i = \max\left\{\sqrt{a_i^2 + a_i r_i + \frac{c_i^2}{4}} + \frac{c_i}{2}, \sqrt{a_i^2 + a_i c_i + \frac{r_i^2}{4}} + \frac{r_i}{2}\right\}$$

for $i = 1,2,\cdots,n$, where if one of the numbers in the minimum is not real, we omit it.

□

(h) For the condition number:

If A is diagonally dominant then:

$$\max_i R_i / n^{1/2} \min_i R_i \leq c(A) \quad . \tag{17}$$

Proof: Since A is diagonally dominant the Lévy-Desplanques Theorem implies that $\det A$ is nonzero. Thus, $\sigma_n > 0$. Now (17) is clear from 0 (16) and 0 (17).

□

§6:4

In this section we shall give bounds which involve only row sums and $\operatorname{tr} AA^*$.

(e) For sum of eigenvalues:

Define,

$$r_i = \sum_j a_{ij} \quad , \quad c_i = \sum_j a_{ji} \quad ,$$

$$p = \frac{1}{n}\Big(\sum_{i<k} |c_i - c_k|^2\Big)^{1/2} \quad \text{and} \quad q = \frac{1}{n}\Big(\sum_{i<k} |r_i - r_k|^2\Big)^{1/2} \quad .$$

For any matrix A ,

$$\sum_i |\lambda_i|^2 \le ||A||^2 - (p-q)^2 \quad . \tag{1}$$

In fact $p = ||PAQ||$ and $q = ||QAP||$, where, $P = \frac{1}{n} J$, $Q = I-P$ and all the elements of J are one. Further, if $pq \neq 0$ then equality holds in (1) if and only if,

$$C_o = PAP + QAQ + \Big(\frac{q}{p}\Big)^{1/2} PAQ + \Big(\frac{p}{q}\Big)^{1/2} QAP$$

is normal while if $pq = 0$ equality holds in (1) if and only if, $C_o = PAP + QAQ$ is normal.

<u>Proof</u>: See [25].

□

(f) For product of eigenvalues:

Let $a_{ii} = 1$, $P_i < 1$, $i = 1,2,\cdots,n$,

$$p = \max_i P_i \quad , \quad P = \sum_i P_i \quad ,$$

$$t = \operatorname{tr} A^* A - n \quad \text{and} \quad q = \max_{i \neq j} |a_{ij}| \quad .$$

Then:

$$e^{qP/p^2}(1-p)^{qP/p^2} \leq e^{t/p}(1-p)^{t/p^2} \leq |\det A| \quad .$$

Proof: See [7, pg. 72]. □

§6:5

Given the row sums R_i , and $||AA^* - A^*A||$ we have the following bounds for the sum of eigenvalues:

(e) For sum of eigenvalues:

Given $D = AA^* - A^*A$, we get

$$\sum_i R_i^2/n - \left(\frac{n^3-n}{12}\right)^{1/2} ||D|| \le \sum_i |\lambda_i|^2$$

$$\le \left(\left(\sum_i R_i^2\right)^2 - \frac{1}{2}||D||^2\right)^{1/2} . \qquad (1)$$

Proof: Inequality (1) is immediate from 0 (20), 5:4 (1) and 5:4 (3).

□

CHAPTER 7

DETERMINANT

§7:0 Preliminaries.

Given A, the determinant of A, written $\det A$, is defined as,

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \prod_i a_{\sigma(i)i} \quad , \tag{1}$$

where S_n, is the symmetric group of order n and $\operatorname{sgn} \sigma$ is one if σ is an even permutation and negative one if σ is an odd permutation. It can be shown that $\det(A-\lambda I) = (-1)^n\lambda^n + (-1)^{n-1}c_1\lambda^{n-1} + (-1)^{n-2}c_2\lambda^{n-2} + \cdots + c_n$, where c_r is the sum of all the principle minors of order r of A, $1 \leq r \leq n$ (e.g. see [26, pg. 54]). Thus, taking $\lambda = 0$, we get

$$\det A = c_n = \lambda_1 \lambda_2 \cdots \lambda_n \quad , \tag{2}$$

since c_r is also the sum of all products of the n eigenvalues of A taken r at a time. Alternatively, since the determinant of a triangular matrix is the product of its diagonal elements, and since similar matrices have the same determinant, (2) also follows from Schur's triangularization Theorem.

Now, we give the results which involve only $\det A$.

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

Given A,

$$|\det A|^{1/n} \leq |\lambda_1| \quad . \tag{3}$$

When A is normal, equality holds if and only if $A = cU$, for some scalar c and unitary U.

Proof: Inequality (3) is immediate from (2). The equality conditions follow from Theorem 11:0 (6).

□

(b) For $|\lambda_n| = \min_i |\lambda_i|$:

Given A,

$$|\lambda_n| \leq |\det A|^{1/n} \quad . \tag{4}$$

When A is normal, equality holds if and only if $A = cU$, for some scalar c and unitary U.

Proof: Inequality (4) is clear. The equality conditions follow from Theorem 11:0 (6).

□

(e) For sum of eigenvalues:

Given A,

$$n\,|\det A|^{\alpha/n} \leq \sum_i |\lambda_i|^{\alpha} \quad , \quad \alpha \geq 0 \quad . \tag{5}$$

For normal A, equality holds if and only if $A = cU$ for some scalar c and unitary U.

Proof: Inequality (5) follows by an application of the arithmetic - geometric mean inequality to, $|\det A|^\alpha = |\lambda_1|^\alpha |\lambda_2|^\alpha \cdots |\lambda_n|^\alpha$, $\alpha \geq 0$. Equality holds in (5) if and only if $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n|$, which for normal A happens if and only if $A = cU$, for some scalar c and unitary U (see Theorem 11:0 (6)).

□

(f) For product of eigenvalues:

Given A ,

$$\lambda_1 \lambda_2 \cdots \lambda_n = \det A \quad . \tag{6}$$

□

(g) For singular values:

Given A ,

$$|\det A|^{1/n} \leq \sigma_1 \quad ; \tag{7}$$

$$\sigma_n \leq |\det A|^{1/n} \quad ; \tag{8}$$

$$\sigma_1 \sigma_2 \cdots \sigma_n = |\det A| \; ; \tag{9}$$

$$n \; |\det A|^{\alpha/n} \leq \sum_i \sigma_i{}^\alpha \quad , \quad \alpha \geq 0 \quad . \tag{10}$$

Equality holds in (7) if and only if equality holds in (8) if and only if equality holds in (10) if and only if $A = cU$ for some nonnegative c and unitary U .

Proof: From Theorem II:4:1 (1) we have $|\det A| = \sigma_1\sigma_2\cdots\sigma_n$. Now, inequalities (7) and (8) are clear. Inequality (10) follows from the arithmetic - geometric mean inequality. Further, equality in (7) or (8) holds if and only if,

$$\sigma_1 = \sigma_2 = \cdots = \sigma_n = |\det A|^{1/n} \quad . \tag{11}$$

Further, equality in (10) holds if and only if (11) holds. However, if (11) holds, then from the Singular value decomposition theorem (see Theorem II:4:0 (4)), $A = \sigma_1 UV$. The converse is trivial.

□

§7:1

If $\det A$ and $\operatorname{tr} A$ are known, then results of Section 0 and 1:0 can be combined to yield improved bounds for the eigenvalues and the singular values of A. In addition the following lower bound for $c(A)$ holds:

(h) For the condition number:

If $\det A \neq 0$ then:

$$\frac{|\operatorname{tr} A|}{n\,|\det A|^{1/n}} \leq c(A) \quad . \tag{1}$$

For normal A, equality holds in (1) if and only if A is a scalar matrix.

<u>Proof</u>: Inequality (1) is clear from 0:(8) and 1:0 (7). Further, for normal A, if equality holds in (1) then necessarily $|\lambda_1| = |\operatorname{tr} A| / n$, which implies $\lambda_1 = \cdots = \lambda_n$. Now from the Theorem 1:0 (7), A must be a scalar. The converse is clear.

□

§7:2

Given det A and the diagonal elements of A, results of Sections 0, 1, 1:0 and 2:0 hold. In addition the following result for the condition number holds:

(h) For the condition number:

If $\det A \neq 0$ then, $\sigma_n > 0$ and

$$1 \leq \frac{n \max\limits_i |a_{ii}|}{|\mathrm{tr}\, A|} \leq \frac{|\lambda_1|}{|\lambda_n|} \leq c(A) \quad , \tag{1}$$

provided $\mathrm{tr}\, A \neq 0$.

Proof: Since $\det A = |\sigma_1 \cdots \sigma_n|$ (see Theorem II:4:1 (1)), $\det A \neq 0$ implies $\sigma_n > 0$. Now, (1) follows from 1:0 (4) and 2:0 (3).

□

§7:3

If det A and tr A^2 are known then the results of Section 0 and 3:0 hold. In addition if det A is non-zero, we have the following:

(h) For the condition number:

If $\det A \neq 0$, then:

$$\frac{(|\mathrm{tr}\ A^2|)^{1/2}}{n^{1/2}\ |\det A|^{1/n}} \le \frac{|\lambda_1|}{|\lambda_n|} \le c(A) \quad . \tag{1}$$

<u>Proof</u>: The proof is immediate from 3:0 (2) and 0 (4).

□

§7:4

In this section we shall give results which involve $\det A$ and $\operatorname{tr} AA^*$. Their proofs involve the use of the arithmetic - geometric mean inequality. Wittmeyer and Wegner used the same idea to derive bounds for a real, positive definite matrix (see [7, pg. 69]).

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

Given A,

$$\left(\frac{n-1}{\operatorname{tr} AA^* - |\det A|^{2/n}}\right)^{n-1} |\det A|^2 \le |\lambda_1|^2$$
$$\le \operatorname{tr} AA^* - (n-1)\left(\frac{|\det A|^2}{\operatorname{tr} AA^*}\right)^{\frac{1}{n-1}} . \qquad (1)$$

<u>Proof</u>: Applying the arithmetic - geometric mean inequality to $|\lambda_2|^2, |\lambda_3|^2, \cdots, |\lambda_n|^2$, we have,

$$|\det A|^2 \le |\lambda_1|^2 \left(\frac{\sum_j |\lambda_j|^2 - |\lambda_1|^2}{n-1}\right)^{n-1} . \qquad (2)$$

Now $\sum_i |\lambda_i|^2 \le \operatorname{tr} AA^*$ implies the inequality on the left, as $|\det A|^{2/n} \le |\lambda_1|^2$. Also, solving for the second $|\lambda_1|^2$ in (2), we obtain,

$$|\lambda_1|^2 \le \sum_j |\lambda_j|^2 - (n-1)\left(\frac{|\det A|^2}{|\lambda_1|^2}\right)^{1/(n-1)} , \qquad (3)$$

which yields the inequality on the right, since $|\lambda_1|^2 \le \sum_i |\lambda_i|^2 \le \operatorname{tr} AA^*$.

□

(b) For $|\lambda_n| = \min |\lambda_i|$:

Given A ,

$$\left(\frac{n-1}{\operatorname{tr} AA^*}\right)^{n-1} |\det A|^2 \leq |\lambda_n|^2 \leq \operatorname{tr} AA^* - (n-1)\, |\det A|^{2/n} \quad . \tag{4}$$

<u>Proof</u>: If $\lambda_n = 0$ then (4) is clear. In case $\lambda_n \neq 0$, applying the arithmetic - geometric mean inequality to $|\lambda_1|^2, |\lambda_2|^2, \cdots, |\lambda_{n-1}|^2$, we have,

$$|\det A|^2 \leq |\lambda_n|^2 \left(\frac{\sum_i |\lambda_i|^2 - |\lambda_n|^2}{n-1}\right)^{n-1} \leq |\lambda_n|^2 \left(\frac{\operatorname{tr} AA^*}{n-1}\right)^{n-1} ,$$

which establishes the left-hand-side inequality. Now, if $\lambda_n = 0$, then the inequality on the right holds. Let $\lambda_n \neq 0$. Then

$$|\lambda_n|^2 \leq \sum_i |\lambda_i|^2 - (n-1)\left(\frac{|\det A|^2}{|\lambda_n|^2}\right)^{1/n-1} ,$$

and $|\lambda_n|^2 \leq |\det A|^{2/n}$, yields the right-hand-side of (4).

□

(c) For $|\lambda_k|$:

Given A , for $1 \leq k \leq n$,

$$|\lambda_k|^2 \leq \operatorname{tr} AA^* - (n-1)\left(\frac{k\, |\det A|^2}{\operatorname{tr} AA^*}\right)^{1/n-1} \quad . \tag{5}$$

<u>Proof</u>: As in the proof of inequality (4), the arithmetic - geometric mean inequality gives,

$$|\lambda_k|^2 \le \sum_i |\lambda_i|^2 - (n-1)\left(\frac{|\det A|^2}{|\lambda_k|^2}\right)^{1/n-1} .$$

Now using $\sum_i |\lambda_i|^2 \le \operatorname{tr} AA^*$, we get $|\lambda_k|^2 \le \operatorname{tr} AA^* / k$, which together with the above inequality proves (5).

□

(d) For the spread:

Given A,

$$\left\{\left(\frac{n-1}{\operatorname{tr} AA^* - |\det A|^{2/n}}\right)^{n-1} |\det A|^2\right\}^{1/2}$$

$$- \{\operatorname{tr} AA^* - (n-1)\ |\det A|^{2/n}\}^{1/2} \le \operatorname{sp}(A) . \tag{6}$$

<u>Proof</u>: Inequality (6) is immediate from (1) and (4).

□

(g) For singular values:

Given A,

$$\left(\frac{n-1}{\operatorname{tr} AA^* - |\det A|^{2/n}}\right)^{n-1} |\det A|^2 \le \sigma_1{}^2$$

$$\le \operatorname{tr} AA^* - (n-1)\left(\frac{|\det A|^2}{\operatorname{tr} AA^*}\right)^{1/n-1} ; \tag{7}$$

$$\left(\frac{n-1}{\operatorname{tr} AA^*}\right)^{n-1} |\det A|^2 \le \sigma_n{}^2 \le \operatorname{tr} AA^* - (n-1)\ |\det A|^{2/n} ; \tag{8}$$

and,

$$|\sigma_k|^2 \le \operatorname{tr} AA^* - (n-1)\left(\frac{k\ |\det A|^2}{\operatorname{tr} AA^*}\right)^{1/n-1} . \tag{9}$$

Proof: Since $|\det A| = \sigma_1 \sigma_2 \cdots \sigma_n$ and $\sum_i \sigma_i^2 = \operatorname{tr} AA^*$, the above inequalities can be proved with the arguments similar to those used to prove (1), (4) and (5).

□

(h) For the condition number:

Given A, if $\det A \neq 0$, then:

$$\frac{(n-1)^{n-1}}{(\operatorname{tr} AA^* - |\det A|^{2/n})^{n-1} p} \leq c^2(A)$$

$$\leq \frac{(\operatorname{tr} AA^*)^{n-1} \{\operatorname{tr} AA^* - (n-1)(\frac{|\det A|^2}{\operatorname{tr} AA^*})^{1/n-1}\}}{(n-1)^{n-1} |\det A|^2} ,$$

where,

$$p = \min(\operatorname{tr} AA^*/n \, , \, \operatorname{tr} AA^* - (n-1) \, |\det A|^{2/n}) \; .$$

Proof: Proof follows from the inequalities (1), (4) and the fact that $\sigma_n^2 \leq \operatorname{tr} AA^* / n$.

□

§7:5

Given $\det A$ and $||AA^* - A^*A||$ we have the following result:

(h) For the condition number:

Given A, if $|\det A| \neq 0$, then:

$$\frac{||AA^* - A^*A||}{2^{1/2} n |\det A|^{2/n}} \leq c^2(A) \quad . \tag{1}$$

<u>Proof</u>: Inequality (1) is immediate from 5:0 (2) and 7:0 (8).

□

§7:5.4

In this section we shall give results which involve $\det A$, $\operatorname{tr} AA^*$ and $||AA^* - A^*A||$. Their proofs run along the same lines as of those proved in 7:4, except that now we use the inequality 5:4 (1), which says,

$$\sum_i |\lambda_i|^2 \leq q = (||A||^4 - \tfrac{1}{2}\,||AA^* - A^*A||)^{1/2} \quad , \tag{1}$$

instead of $\sum_i |\lambda_i|^2 \leq ||A||^2$. We shall thus omit these proofs.

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

With q as given by (1),

$$\left(\frac{n-1}{q - |\det A|^{2/n}}\right)^{n-1} |\det A|^2 \leq |\lambda_1|^2$$

$$\leq q - (n-1)\left(\frac{|\det A|^2}{q}\right)^{1/n-1} \quad . \tag{2}$$

□

(b) For $|\lambda_n| = \min_i |\lambda_i|$.

With q as in (1),

$$\left(\frac{(n-1)}{q}\right)^{n-1} |\det A|^2 \leq |\lambda_n| \leq q - (n-1)\ |\det A|^{2/n} \quad . \tag{3}$$

□

(c) For $|\lambda_k|$:

Given A ,

$$|\lambda_k|^2 \leq q - (n-1)\left(\frac{k\ |\det A|^2}{q}\right)^{1/n-1} \quad , \ k = 1,2,\cdots,n \quad , \qquad (4)$$

where q is as defined by (1).

□

(d) For the spread:

Given A and q as above,

$$\left\{\frac{(n-1)}{(q - |\det A|^{2/n})}\right\}^{1/2} - \{q - (n-1)\ |\det A|^{2/n}\}^{1/2} \leq sp(A) \ . \qquad (5)$$

□

§7:6

Given the row sums,

$$R_i = \sum_j |a_{ij}| \quad , \tag{1}$$

we have

$$\sum_i |\lambda_i|^2 \leq \gamma = \sum_i R_i^{\,2} - (p-q)^2 \quad , \tag{2}$$

where, $p = \frac{1}{n}\left(\sum_{i<k} |c_i - c_k|^2\right)$, $q = \frac{1}{n}\left(\sum_{i<k} |r_i - r_k|^2\right)^{1/2}$, $c_i = \sum_j a_{ji}$ and $r_i = \sum_j a_{ij}$ (see 6:0 (12)). Thus, all the results of 7:4 hold with $\operatorname{tr} AA^*$ replaced by γ as defined by (2). In addition we have the following:

(h) For the condition number:

Given A , if $\det A \neq 0$, then:

$$\frac{\max_i R_i}{n^{1/2}\,|\det A|^{1/n}} \leq c(A) \quad .$$

<u>Proof</u>: The proof is clear from 6:0 (16) and 7:0 (8) .

□

CHAPTER 8

REAL EIGENVALUES

§8:0 Preliminaries.

Given the matrix A, its eigenvalues need not be real. Even if A is real, all its eigenvalues can be complex. However, there exist matrices, which have real eigenvalues. For example, if the order n of a real matrix A, is odd then at least one of its eigenvalues is real. Also, it is well-known that all the eigenvalues of a Hermitian matrix are real.

Below, we state some necessary and sufficient conditions for $m(\leq n)$ eigenvalues of A to be real. The proofs can be found in [12].

Theorem 1: If A has m distinct real eigenvalues, then there is a positive semidefinite matrix S of rank m such that $AS = SA^*$; conversely, given any such S, then A must have at least m real eigenvalues, and if $m = n$, then A is diagonalizable. □

Theorem 2: Given $A = (a_{ij})$, let $r_i = \sum_k a_{ik}$, $i = 1,2,\cdots,n$. Define an $n\times n$ matrix $E = (e_{ij})$, $e_{ij} = r_i - \overline{r}_j$ and suppose that

$$A^* - A = cE \quad ,$$

for some real $c > -\frac{1}{n}$ (or $c \geq -\frac{1}{n}$ if A has real trace or real non-zero determinant). Then A has all its eigenvalues real (and A is diagonalizable if $c \neq -\frac{1}{n}$). □

Since the eigenvalues λ_i , $i = 1,2,\cdots,n$, of A are given to be real in this chapter, we shall always assume (unless otherwise stated) that they are ordered as:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad . \tag{1}$$

§8:1

In this section we shall give bounds for eigenvalues which involve only tr A and n .

(a) For $\lambda_1 = \max_i \lambda_i$:

Define,

$$m = \text{tr } A / n \quad . \tag{1}$$

Then

$$m \leq \lambda_1 \quad . \tag{2}$$

Equality holds in (2) if and only if:

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n \quad . \tag{3}$$

<u>Proof</u>: Trivial.

□

(b) For $\lambda_n = \min_i \lambda_i$:

Given m as in (1),

$$\lambda_n \leq m \quad . \tag{4}$$

Equality holds in (4) if and only if (3) holds.

<u>Proof</u>: Trivial.

□

§8:3

Let the eigenvalues of A be ordered as

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \quad .$$

Given $\operatorname{tr} A^2$, using the fact that the eigenvalues of A are real we obtain the following results:

(c) For $|\lambda_k|$:

Given $\operatorname{tr} A^2$,

$$|\lambda_k| \leq (\operatorname{tr} A^2 / k)^{1/2} \quad , \quad k = 1,2,\cdots,n \quad . \tag{1}$$

Proof: Trivial.

□

(f) For product of eigenvalues:

Given $\operatorname{tr} A^2$,

$$(\det A)^{2/n} \leq \operatorname{tr} A^2 / n \quad . \tag{2}$$

Equality holds in (2) if and only if:

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n \quad . \tag{3}$$

Proof: The proof follows at once by an application of the arithmetic - geometric mean inequality to λ_i^2 , $i = 1,2,\cdots,n$.

□

(g) For singular values:

Given $\mathrm{tr}\, A^2$,

$$\sigma_n \le |\lambda_n| \le (\mathrm{tr}\, A^2 / n)^{1/2} \quad ; \tag{4}$$

$$(\sigma_1 \sigma_2 \cdots \sigma_n)^{2/n} \le \mathrm{tr}\, A^2 / n \quad ; \tag{5}$$

$$\mathrm{tr}\, A^2 \le \sum_i \sigma_i^2 \quad . \tag{6}$$

<u>Proof</u>: Inequalities (4) and (5) follow from Theorem II:4:1 (1) and the inequalities (1) and (2) respectively. Inequality (6) follows using Theorem II:4:1 (1) and the fact that $\sum_k \lambda_i^2 = \mathrm{tr}\, A^2$. □

§8:3.1

Many localization results can be obtained using the tr A, tr A^2 and the fact that the eigenvalues of A are real. A family of such results is derived in [37] and [63].

All the results to follow are proved in [37] and [63]. We shall omit their proofs. We shall need the following notation:

$$m = \mathrm{tr}\, A / n \quad \text{and} \quad s^2 = \mathrm{tr}\, A^2 / n - m^2 \quad . \tag{1}$$

(a) For $\lambda_1 = \max_i \lambda_i$:

With m and s as in (1),

$$m + s(n-1)^{-1/2} \le \lambda_1 \le m + s(n-1)^{1/2} \quad . \tag{2}$$

Equality holds on the left (right) if and only if the $(n-1)$ largest (smallest) eigenvalues of A are equal.

Proof: See [63].

□

(b) For $\lambda_n = \min_i \lambda_i$:

With m and s as in (1)

$$m - s(n-1)^{1/2} \le \lambda_n \le m - s/(n-1)^{1/2} \quad . \tag{3}$$

Equality holds on the left (right) of (3) if and only if equality holds on the left (right) of (2).

Proof: See [63].

□

(c) For λ_k :

Given A ,

$$m - s\left(\frac{k-1}{n-k+1}\right)^{1/2} \leq \lambda_k \leq m + s\left(\frac{n-k}{k}\right)^{1/2} \quad . \tag{4}$$

Equality holds on the left of (4) if and only if

$$\lambda_1 = \cdots = \lambda_{k-1} \quad \text{and} \quad \lambda_k = \cdots = \lambda_n \tag{5}$$

and on the right of (4) if and only if

$$\lambda_1 = \cdots = \lambda_k \quad \text{and} \quad \lambda_{k+1} = \cdots = \lambda_n \quad . \tag{6}$$

<u>Proof</u>: See [63].

□

(d) For the spread:

Let s be given by (1). Then:

$$2s \leq \operatorname{sp}(A) \leq (2n)^{1/2} s \quad . \tag{7}$$

When $n > 2$, equality holds on the right if and only if

$$\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} = \frac{1}{2}(\lambda_1 + \lambda_n) \quad . \tag{8}$$

In case n is even, say $n = 2q$, then equality holds on the left of (7) if and only if

$$\lambda_1 = \lambda_2 = \cdots = \lambda_q \quad \text{and} \quad \lambda_{q+1} = \lambda_{q+2} = \cdots = \lambda_n \quad . \tag{9}$$

Furthermore, if n is odd, say $n = 2q+1$, then:

$$2s\ n/(n^2-1)^{1/2} \leq sp(A) \quad . \tag{10}$$

Equality holds in (10) if and only if (9) is true.

<u>Proof</u>: See [63]. □

(e) For sum of eigenvalues:

(i) If we define,

$$\lambda_{(k,\ell)} = \frac{1}{\ell-k+1} \sum_{k}^{\ell} \lambda_j \quad , \tag{11}$$

and m and s^2 as in (1), then:

$$m - s\left(\frac{k-1}{n-k+1}\right)^{1/2} \leq \lambda_{(k,\ell)} \leq m + s\left(\frac{n-\ell}{\ell}\right)^{1/2} \quad . \tag{12}$$

For $(k,\ell) = (1,n)$, (12) collapses and equality always holds. When $(k,\ell) \neq (1,n)$ equality holds on the left of (12) if and only if

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} \quad \text{and} \quad \lambda_k = \lambda_{k+1} = \cdots = \lambda_n \quad , \tag{13}$$

and on the right if and only if

$$\lambda_1 = \lambda_2 = \cdots = \lambda_\ell \quad \text{and} \quad \lambda_{\ell+1} = \lambda_{\ell+2} = \cdots = \lambda_n \quad . \tag{14}$$

<u>Proof</u>: See [63]. □

(ii) If $k = 1$ or $\ell = n$, the bounds for $\lambda_{(k,\ell)}$ can be improved:

$$\lambda_{(1,\ell)} \geq \begin{cases} m + \dfrac{s}{(n-1)^{1/2}} & \text{if } \ell \leq \dfrac{n}{2} \\ \\ m + \dfrac{s(n-\ell)}{(n-1)^{1/2}} & \text{if } \ell \geq \dfrac{n}{2} \end{cases} \tag{15}$$

$$\lambda_{(k,n)} \geq \begin{cases} m - \dfrac{s(k-1)}{(n-k+1)(n-1)^{1/2}} & \text{if } k \leq \dfrac{n}{2}+1 \\ \\ m - \dfrac{s}{(n-1)^{1/2}} & \text{if } k \geq \dfrac{n}{2}+1 \ . \end{cases} \tag{16}$$

Equality holds in (15) if and only if

$$\begin{aligned} &\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} && \text{when } \ell < \tfrac{n}{2} \\ \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} \ \text{or}\ &\lambda_2 = \lambda_3 = \cdots = \lambda_n && \text{when } \ell = \tfrac{n}{2} \\ &\lambda_2 = \lambda_3 = \cdots = \lambda_n && \text{when } \ell > \tfrac{n}{2} \ . \end{aligned} \tag{17}$$

Equality holds in (16) if and only if

$$\begin{aligned} &\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} && \text{when } k < \tfrac{n}{2} + 1 \\ \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} \ \text{or}\ &\lambda_2 = \cdots = \lambda_n && \text{when } k = \tfrac{n}{2} +1 \\ &\lambda_2 = \lambda_3 = \cdots = \lambda_n && \text{when } k > \tfrac{n}{2} +1 \ . \end{aligned} \tag{18}$$

<u>Proof</u>: See [63].

□

(iii) With m and s^2 as before,

$$\lambda_k - \lambda_\ell \leq s\, n^{1/2} \left(\frac{1}{k} + \frac{1}{n-\ell+1}\right)^{1/2} \quad , \quad 1 \leq k < \ell \leq n \quad . \tag{19}$$

Equality holds if and only if,

$$\begin{gathered} \lambda_1 = \lambda_2 = \cdots = \lambda_k \\ \lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_{\ell-1} = \operatorname{tr} A / n \\ \lambda_\ell = \lambda_{\ell+1} = \cdots = \lambda_n \quad . \end{gathered} \tag{20}$$

Proof: See [63]. □

(f) For ratios of eigenvalues:

(i) Define,

$$\gamma_{k\ell} = \lambda_k / \lambda_\ell \quad , \quad 1 \leq k < \ell \leq n \quad , \tag{21}$$

and

$$c = \frac{(\operatorname{tr} A)^2}{\operatorname{tr} A^2} - (\ell-1) \quad . \tag{22}$$

If $\operatorname{tr} A \geq 0$ and

$$(\ell-1) \operatorname{tr} A^2 < (\operatorname{tr} A)^2 \quad , \tag{23}$$

then:

$$\lambda_\ell > 0 \quad ,$$

and

$$\gamma_{k\ell} \le \frac{c+k+\{\frac{n-\ell+1}{k}\,(c+k)(n-\ell+1-c)\}^{1/2}}{c+k-\{\frac{k}{n-\ell+1}\,(c+k)(n-\ell+1-c)\}^{1/2}} \quad . \tag{24}$$

Further, equality holds in (24) if and only if

$$\begin{gathered} \lambda_1 = \lambda_2 = \cdots = \lambda_k \\ \lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_{\ell-1} = \operatorname{tr} A^2 / \operatorname{tr} A \quad , \\ \lambda_\ell = \lambda_{\ell+1} = \cdots = \lambda_n \quad . \end{gathered} \tag{25}$$

Proof: See [37]. □

Remark 1. For $\ell = k+1$, (24) yields:

$$\lambda_k / \lambda_{k+1} \le \frac{m + s(\frac{n-k}{k})^{1/2}}{m - s/(\frac{n-k}{k})^{1/2}} \quad , \tag{26}$$

with equality if and only if

$$\lambda_1 = \lambda_2 = \cdots = \lambda_k \quad \text{and} \quad \lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n \quad .$$ □

(ii) Define,

$$\delta_{k\ell} = \frac{\lambda_k - \lambda_\ell}{\lambda_k + \lambda_\ell} \quad , \qquad 1 \le k < \ell \le n \quad . \tag{27}$$

If $\operatorname{tr} A \ge 0$ and for $t = \max(k, 2\ell-n-1)$,

$$(t-1) \operatorname{tr} A^2 < (\operatorname{tr} A)^2 \quad . \tag{28}$$

Then:

$$\lambda_k + \lambda_\ell > 0 ,$$

and

$$\delta_{k\ell} \leq \frac{\{(c+k)(n-\ell+1-c)\}^{1/2} \quad (n-\ell+1+k)}{2(c+k)\{k(n-\ell+1)\}^{1/2} + \{(c+k)(n-\ell+1-c)\}^{1/2}(n-\ell+1-k)} , \quad (29)$$

where c is given by (22). Equality holds if and only if (25) holds.

<u>Proof</u>: See [37].

□

(iii) Given $k > 1$ (or $\ell < n$) the best lower bound for $\delta_{k\ell}$ is 0. If $k = 1$ and $\ell = n$ then for $\mathrm{tr}\, A > 0$, we have:

$$\frac{ns}{2m(n-1)^{1/2} + s(n-2)} \leq \delta_{1n} , \quad (30)$$

where m and s are as given by (1). Equality holds if and only if

$$\lambda_2 = \lambda_3 = \cdots = \lambda_n .$$

<u>Proof</u>: See [37].

□

(iv) With

$$\eta_{k\ell} = \frac{(k\,\lambda_k + (n-\ell+1)\,\lambda_\ell)^2}{k\,\lambda_k^{\,2} + (n-\ell+1)\,\lambda_\ell^{\,2}} , \quad 1 \leq k < \ell \leq n \quad (31)$$

if $\mathrm{tr}\, A \geq 0$ and $k\,\lambda_k + (n-\ell+1)\,\lambda_\ell > 0$, then:

$$\frac{(\mathrm{tr}\ A)^2}{\mathrm{tr}\ A^2} - (\ell-k-1) \le \eta_{k\ell} \quad . \tag{32}$$

Equality holds if and only if (25) holds.

Proof: See [37]. □

(v) Given A ,

$$\frac{(\mathrm{tr}\ A)^2}{\mathrm{tr}\ A^2} - n + 2 \le \frac{(\lambda_1+\lambda_n)^2}{\lambda_1{}^2+\lambda_n{}^2} \quad . \tag{33}$$

For $n > 2$, equality holds if and only if $\lambda_1 + \lambda_n \neq 0$ and

$$\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} = \frac{\lambda_1{}^2 + \lambda_n{}^2}{\lambda_1 + \lambda_n} \quad .$$

Proof: See [63]. □

(g) For singular values:

Given m and s^2 as in (1),

$$m + s / (n-1)^{1/2} \le \sigma_1 \quad . \tag{34}$$

When

$$\mathrm{tr}\ A \ge 0 \quad \text{and} \quad (n-1)\ \mathrm{tr}\ A^2 < (\mathrm{tr}\ A)^2 \quad , \tag{35}$$

then:

$$\sigma_n \le m - s / (n-1)^{1/2} \quad . \tag{36}$$

Further, if A is Hermitian then $\lambda_n > 0$ implies that A is positive definite. Also, then equality holds in (34) if and only if,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} \quad ,$$

and equality holds in (36) if and only if

$$\lambda_2 = \lambda_3 = \cdots = \lambda_n \quad .$$

Proof: Inequality (34) follows from (2) and Theorem II:4:1 (1). Also the inequalities (3) and (35) imply that $\lambda_n > 0$. Thus, from Theorem II:4:1 (1), $\sigma_n \leq \lambda_n$ and (36) follows from (3). Finally, if A is Hermitian then $\lambda_n > 0$ implies that A is positive definite, which in turn implies that, $\sigma_i = \lambda_i$, $i = 1,2,\cdots,n$ (see 13:0 (11)). Thus equality conditions follow from the equality conditions in the inequalities (2) and (3).

□

(h) For the condition number:

If $\operatorname{tr} A \geq 0$ and $c > 0$, where

$$c = \frac{(\operatorname{tr} A)^2}{\operatorname{tr} A^2} - (n-1) \quad ,$$

then:

$$0 < \frac{m + s / (n-1)^{1/2}}{m - s / (n-1)^{1/2}} \leq c(A) \quad . \tag{37}$$

In case A is Hermitian, equality holds in (37) if and only if A is a positive scalar matrix.

Proof: The proof is immediate from inequalities (34) and (36). $\square$

§8:4

Given $\operatorname{tr} AA^*$ and that all the eigenvalues of A are real, in addition to the results of 4:0, we have the following result:

(e) For sum of eigenvalues:

Given $\operatorname{tr} AA^*$,

$$\operatorname{tr} A^2 = \sum_i \lambda_i^2 \leq \operatorname{tr} AA^* \quad . \tag{1}$$

Equality holds if and only if A is Hermitian.

Proof: Since the eigenvalues are real, (1) is clear from 4:0 (9). Also, we have that equality in (1) holds if and only if A is normal. Thus, the result will follow, if we can prove that a normal matrix has real eigenvalues if and only if it is Hermitian. Sufficiency is clear. To prove the necessity, we write $A = UDU^*$, where U is unitary and $D = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ with real eigenvalues λ_i $i = 1, 2, \cdots, n$. Then we have, $A^* = (UDU^*)^* = A$, which proves that A is Hermitian.

□

CHAPTER 9

REAL MATRIX

§9:0 Preliminaries.

Given real A, its characteristic polynomial $\det(A-\lambda I)$ has real coefficients. Thus complex eigenvalues of a real matrix occur in conjugate pairs and, if the order n of A is odd then it has at least one real eigenvalue. Further, if n is even and $\det A < 0$ then A has at least two real eigenvalues. The following theorem provides a necessary and sufficient condition for a real matrix to have real eigenvalues:

Theorem 1: Let A be a real $n \times n$ matrix, with eigenvalues, λ_i, $i = 1,2,\cdots,n$. Then there exists a real orthogonal matrix U and a triangular matrix T such that $A = UTU'$, with $t_{ii} = \lambda_i$, $i = 1,2,\cdots,n$ if and only if the eigenvalues of A are real.

Proof: (See [32, pg. 497].

□

In this chapter we shall consider a general real matrix. However, several special types of real matrices exist. Nonnegative matrices are dealt with in the next chapter. Unless, otherwise stated we shall assume that the eigenvalues are ordered as:

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \quad .$$

§9:4.2

Given $\operatorname{tr} AA^*$ and the diagonal elements of a real matrix, we have the following results:

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

Given real A ,

$$|\lambda_1| \le \max_i |a_{ii}| + (\operatorname{tr} AA^* - \sum_i a_{ii}^2)^{1/2} \quad . \tag{1}$$

Proof: Since $|\lambda_1| \le \sigma_1$, the proof follows from (3) below. □

(b) For $|\lambda_n| = \min_i |\lambda_i|$:

Given real A ,

$$\min_i |a_{ii}| - (\operatorname{tr} AA^* - \sum_i a_{ii}^2)^{1/2} \le |\lambda_n| \quad . \tag{2}$$

Proof: From Theorem II:4:1 (1), we have $\sigma_n \le |\lambda_n|$. Thus (2) follows from (4) below. □

The following result is given in [7, pg. 67]:

(g) For singular values:

Given real A ,

$$\sigma_1 \leq \max_i a_{ii} + (\text{tr } AA^* - \sum_i a_{ii}^2)^{1/2} \quad ; \tag{3}$$

and

$$\min |a_{ii}| - (\text{tr } AA^* - \sum_i a_{ii}^2)^{1/2} \leq \sigma_n \quad . \tag{4}$$

□

§9:6

The following result involving row sums of a real matrix is given in [54]:

(c) For λ_k :

Let A be a real, irreducible matrix such that $a_{ii} \geq 0$, $1 \leq i \leq n$, $a_{ij} \leq 0$ for all $i \neq j$. Further, assume that A is weakly diagonally dominant. Then $\lambda = 0$ is an eigenvalue of A if and only if

$$\sum_j a_{ij} = 0 \quad , \quad i = 1,2,\cdots,n \quad . \tag{1}$$

Proof: If (1) holds then clearly $\lambda = 0$ is an eigenvalue of A, with corresponding eigenvector $x = (1,1,\cdots,1)'$. Conversely, let x be an eigenvector corresponding to the zero eigenvalue of A and r be one of the indices for which $|x_i|$, $i = 1, 2, \ldots n$ is maximum. Finally, suppose the pth relation in (1) is not an equality. Then we have

$$a_{pp} > \sum_{j \neq p} |a_{pj}| ,$$

and $Ax = 0$ implies,

$$-a_{pp}x_p = \sum_{k \neq p} a_{pk} x_k ,$$

and

$$\begin{aligned} a_{pp} |x_p| &\leq \sum_{k \neq p} |a_{pk}| \, |x_r| \\ &< a_{pp} |x_r| . \end{aligned}$$

Thus $|x_r| > |x_k|$ for at least one value of k and

$$a_{rr} |x_r| \leq \sum_{k \neq r} |a_{rk}| |x_k| ,$$

which gives $a_{rr} < \sum_{k \neq r} |a_{rk}|$, unless $a_{ri} = 0$ for which $|x_r| > |x_i|$. If this, however, is the case, then the rth row contains n-s zeros where s is the number of suffixes j for which $|x_j| = |x_r|$. All the s corresponding rows contain n-s zeros in the same places. But this implies A is reducible. Therefore,

$$|x_1| = |x_2| = \cdots = |x_n| ,$$

and (1) follows. □

§9:6.2

In this section we shall give results which involve row sums and the diagonal elements.

The following result is well-known (e.g. see [15, pg. 261]):

(c) For λ_k :

Let A be a real matrix with positive diagonal elements. If A is weakly diagonally dominant, that is,

$$a_{ii} \geq P_i = \sum_{j \neq i} |a_{ij}| \quad , \quad i = 1,2,\cdots,n \quad , \tag{1}$$

then:

$$\mathrm{Re}(\lambda_i) \geq 0 \quad , \quad i = 1,2,\cdots,n \quad . \tag{2}$$

Further, in case $a_{ii} < 0$, $i = 1,2,\cdots,n$, then:

$$\mathrm{Re}(\lambda_i) \leq 0 \quad , \quad i = 1,2,\cdots,n \quad . \tag{3}$$

If the inequalities in (1) are strict then so are those in (2) and (3).

<u>Proof</u>: From the Gerschgorin's Theorem (see 6:2), for each eigenvalue λ of A , there exists $1 \leq i \leq n$, such that

$$|\lambda - a_{ii}| \leq P_i \quad .$$

Thus we have,

$$|Re(\lambda) - a_{ii}| \leq P_i \quad ,$$

which yields (2) if all the diagonal elements are positive and (3) if they are all negative.

□

(f) For product of eigenvalues:

If the diagonal elements of A are positive and A is diagonally dominant, that is,

$$a_{ii} > \sum_{j \neq i} |a_{ij}| \quad , \quad i = 1,2,\cdots,n \quad ,$$

then $\det A > 0$ and

$$\prod_i (a_{ii}-b_i) \leq \det A \leq \prod_i (a_{ii}+b_i) \quad , \qquad (4)$$

where

$$b_i = \begin{cases} \sum_{i+1}^{n} |a_{ij}| & (1 \leq i < n) \\ 0 & (i = n) \end{cases} .$$

Proof: Since A is diagonally dominant, we conclude that strict inequality holds in (2). Thus all the real eigenvalues are positive. Now, as the complex eigenvalues of A occur in conjugate pairs, we conclude that $\det A > 0$. The proof of (4) is given in [39, pg. 33].

□

CHAPTER 10

NONNEGATIVE MATRIX

§10:0 Preliminaries.

Given a real matrix $A = (a_{ij})$ with $a_{ij} \geq 0$, $i,j = 1,2,\cdots,n$ then A is called nonnegative. In case $a_{ij} > 0$ for all $i,j = 1,2,\cdots,n$, then A is said to be positive. We shall write $A \geq 0$, if A is nonnegative and $A > 0$ if A is positive.

As we shall see, nonnegative matrices have some very interesting spectral properties. Such matrices arise in various applied fields of study. Further, given a matrix A , if we define $|A| = (|a_{ij}|)$ then $\rho(A) \leq \rho(|A|)$, where $\rho(A)$ is the spectral radius of A , that is, $\rho(A) = \max_i |\lambda_i|$. Thus, an upper bound for $\rho(|A|)$ provides an upper bound for $\rho(A)$. In fact if A is nonnegative and $B = (b_{ij})$ is such that $|b_{ij}| \leq a_{ij}$, $i,j = 1,2,\cdots,n$, then $\rho(B) \leq \rho(|B|) \leq \rho(A)$.

In this section we shall first classify the nonnegative matrices and then state some important theorems regarding them.

Definition 1. Given a nonnegative matrix A , it is called primitive if there exists a positive integer k such that $A^k > 0$.

□

Clearly a positive matrix is primitive. We shall denote the (i,j) th element of A^m by $a_{ij}^{(m)}$, where m is a positive integer. The following result provides another definition for an irreducible matrix. It is proved in [31, pg. 122].

<u>Lemma 2</u>. Given a nonnegative matrix A, it is irreducible if and only if for each i,j $(1 \le i, j \le n)$ there exists a positive integer $m = m(i,j)$ such that $a_{ij}^{(m)} > 0$.

□

Given any matrix A, if $a_{ij} \neq 0$, $i \neq j$, $i,j = 1,2,\cdots,n$ then A is irreducible. A matrix with a row or column of zeros is reducible. In fact a reducible matrix must have at least $n-1$ zeros (see [15, pg. 264]). Furthermore, a primitive matrix (in particular a positive matrix) is always irreducible, but the converse is not true. For example,

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

is irreducible but is not primitive (see [31, pg. 123]). However, we have the following result.

<u>Theorem 3</u>. If A is nonnegative, irreducible and $a_{ii} > 0$ for some i, i, $1 \le i \le n$ then A is primitive.

<u>Proof</u>: See [21].

□

We shall further classify nonnegative, irreducible matrices. To accomplish this we shall need the following definitions:

<u>Definition 4</u>: Given indices i,j, $1 \le i, j \le n$,, we say i <u>leads</u> to j, and write $i \to j$, if there exists a positive integer m, such that

$a_{ij}^{(m)} > 0$.

□

<u>Definition 5</u>. If $i \to j$ and $j \to i$ then we say i and j <u>communicate</u>, and write $i \leftrightarrow j$.

□

<u>Definition 6</u>. If $i \leftrightarrow i$, $1 \le i \le n$, then $d(i)$, called the <u>period</u> of the index i , is the greatest common divisor of those k for which $a_{ii}^{(k)} > 0$.

□

<u>Remark 7</u>. Given an index i such that $a_{ii} > 0$, $1 \le i \le n$, then $d(i) = 1$.

□

<u>Lemma 8</u>. Given A nonnegative, if i and j are such that $i \leftrightarrow j$ then $d(i) = d(j)$.

<u>Proof</u>: See [51, pg. 14].

□

<u>Lemma 9</u>. Given a nonnegative, irreducible matrix A , all indices have the same period.

<u>Proof</u>: From Lemma (2) for a nonnegative, irreducible matrix, given indices i and j , $i \leftrightarrow j$. Now the result follows from Lemma (8).

□

Now, in the definition below we classify nonnegative, irreducible matrices:

<u>Definition 10</u>. A nonnegative, irreducible matrix A is called <u>cyclic</u> with period d if the period of any one of its indices satisfies $d > 1$ and is called <u>acyclic</u> if $d = 1$.

□

Below, we give the precise relationship between a nonnegative, irreducible matrix and a primitive matrix.

Theorem 11. A nonnegative, irreducible, acyclic matrix is primitive and conversely.

Proof: See [51, pg. 18]. □

In view of the above discussion, in this chapter we shall give results for nonnegative, nonnegative irreducible, cyclic and acyclic or primitive matrices. We also note that if A is reducible, i.e.

$$A = \begin{pmatrix} D & F \\ 0 & E \end{pmatrix} ,$$

where D and E are $p \times p$ and $q \times q$ complex matrices, then the n eigenvalues of A are the p eigenvalues of D and q eigenvalues of E (see [31, pg. 23]). Thus if A is reducible the problem of locating the eigenvalues of A can be solved by considering irreducible matrices of smaller order.

As we shall see, the largest eigenvalue in modulus of a non-negative matrix is always nonnegative. Usually, this eigenvalue is denoted by r and is often called the Perron root of A. Throughout this chapter we shall assume that the eigenvalues of A are ordered as:

$$r = \lambda_1 \geq |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \quad .$$

The following four theorems are given in Chapter 1 of [51].

Theorem 12. If A is nonnegative, then there exists an eigenvalue r, such that:

(i) r is real and nonnegative;

(ii) r can be associated with nonnegative left and right eigenvectors;

(iii) $r = \lambda_1 \geq |\lambda_i|$, $i = 2,3,\cdots,n$;

(iv) if $0 \leq B \leq A$ (i.e. $0 \leq b_{ij} \leq a_{ij}$, $1 \leq i, j \leq n$) and β is an eigenvalue of B , then $|\beta| \leq r$.

□

Below, we give the Perron - Frobenius Theorem for primitive matrices:

Theorem 13. If A is primitive then, there exists an eigenvalue r such that:

(i) r is real and positive;

(ii) r can be associated with positive left and right eigenvectors;

(iii) $r = \lambda_1 > |\lambda_i|$, $i = 2,3,\cdots,n$;

(iv) r is a simple eigenvalue of A ;

(v) the eigenvectors associated with r are unique to constant multiples;

(vi) If $0 \leq B \leq A$ and β is an eigenvalue of B then $|\beta| \leq r$. Moreover, $\beta = r$ implies $B = A$.

□

Next, we state the Perron - Frobenius Theorem for a nonnegative, irreducible matrix:

Theorem 14. If $A \geq 0$ is irreducible, then all the results of the previous theorem hold except that (iii) is replaced by a weaker result: $r = \lambda_1 \geq |\lambda_i|$, $i = 2,3,\cdots,n$. Further, if A $\lambda_1 = |\lambda_2| = \cdots = |\lambda_\ell|$ then they are the ℓ distinct roots, $\lambda_1 e^{i2\pi k/\ell}$, $k = 0,1,\cdots,\ell-1$, of $\lambda^\ell - r^\ell = 0$.

□

Theorem 15. If A is cyclic with period $d > 1$, then there exist exactly d distinct eigenvalues with $|\lambda| = \lambda_1$. In fact these eigenvalues are $\lambda_1 e^{i2\pi k/d}$, $k = 0,1,\cdots,d-1$.

□

Below, we summarize the eigenvalue inequalities most of which follow from the above theorems.

(c) For λ_k :

(i) Given A nonnegative (primitive) then λ_1 is nonnegative (positive) and

$$|\lambda_k| \leq (<) \quad r = \lambda_1 \quad .$$

Further, λ_1 is simple for A nonnegative, irreducible.

□

(ii) If A is cyclic with period $d > 1$ then there are exactly d distinct eigenvalues with modulus λ_1 , in fact

$$\lambda_{k+1} = \lambda_1 \; e^{i2\pi k/d} \quad , \quad k = 0,1,\cdots,d-1 \; .$$

□

(e) For sum of eigenvalues:

(i) Given a nonnegative matrix A ,

$$0 \leq \text{tr } A^k \quad , \quad k \geq 1 \quad .$$

Proof: Trivial. □

(ii) If A is cyclic then,

$$\text{tr } A = \sum_i \lambda_i = 0 \quad .$$

Proof: The proof is immediate from Theorem (3). □

§10:1

In this section we shall give results which involve only $\operatorname{tr} A$ and n .

(a) For $r = \lambda_1 = \max_i |\lambda_i|$.

(i) Given A nonnegative (primitive),

$$\operatorname{tr} A / n \leq (<) r . \tag{1}$$

<u>Proof</u>: Inequality (1) is immediate from Theorems 0 (12) and 0 (13).

□

(ii) If $A \geq 0$ is irreducible and $\operatorname{tr} A > 0$ then:

$$\frac{\operatorname{tr} A}{n} < r . \tag{2}$$

<u>Proof</u>: From Theorem 0 (3) we have that A is primitive. Now (2) follows from the previous result.

□

(c) For λ_k :

If A is cyclic with period $d > 1$ then:

$$\operatorname{tr} A / n \leq |\lambda_i| \quad , \quad i = 1,2,\cdots,d . \tag{3}$$

<u>Proof</u>: Inequality (3) is immediate from (1) and Theorem 0 (15).

□

(e) For sum of eigenvalues:

The following result is given in [6, pg. 88].

Given A nonnegative and $\operatorname{tr} A$,

$$(\operatorname{tr} A)^m \leq n^{m-1} \sum_i \lambda_i^m = n^{m-1} \operatorname{tr} A^m \quad . \tag{4}$$

Proof: From Holder's inequality we have,

$$(\sum_i a_{ii})^m \leq n^{m-1} \sum_i a_{ii}^m \quad , \quad m \geq 1 \quad .$$

Now, (4) follows from 2 (2) below.

□

(g) For singular values:

Given $A \geq 0$, for any positive integer m ,

$$(\operatorname{tr} A)^m \leq n^{m-1} \sum_i \sigma_i^m \quad . \tag{5}$$

Further, if A is cyclic with period $d > 1$ then

$$d \operatorname{tr} A / n \leq \sum_{i=1}^{d} \sigma_i \quad . \tag{6}$$

Proof: Using Theorem II:4:1 (1), inequalities (5) and (6) follow from (4) and (3) respectively.

□

§10:2

Here we shall give bounds which involve the diagonal elements of A .

(a) For $r = \lambda_1 = \max_i |\lambda_i|$:

The following result, due to Frobenius is well-known:

If $A \geq 0$ iis irreducible, then:

$$\max_i a_{ii} \leq r \quad . \tag{1}$$

Proof: From Theorem 0 (14), there exists a positive vector x such that $Ax = \lambda_1 x$. Thus, we have

$$\lambda_1 x_i = \sum_k a_{ik} x_k \geq a_{ii} x_i \quad , \quad i = 1,2,\cdots,n \quad .$$

Now (1) follows as $x_i > 0$ for $i = 1,2,\cdots,n$.

□

(e) For sum of eigenvalues:

The following result is given in [6, pg. 88].

Given A nonnegative, then for any positive integer k ,

$$\sum_i a_{ii}^{\ k} \leq \operatorname{tr} A^k \quad .$$

Proof: Trivial.

□

§10:3

In this section we shall give bounds which involve $\operatorname{tr} A^2$. Most of the results are immediate from the theorems of Section 0.

(a) For $r = \lambda_1 = \max_i |\lambda_i|$:

If A is nonnegative (primitive) then:

$$(\operatorname{tr} A^2 / n)^{1/2} \leq (<) \ r . \tag{1}$$

□

(c) For λ_k :

If A is cyclic with period $d > 1$ then:

$$\left(\frac{\operatorname{tr} A^2}{n}\right)^{1/2} \leq |\lambda_i| \quad , \quad i = 1,2,\cdots,d . \tag{2}$$

□

(e) For sum of eigenvalues:

The following result is given in [6, pg. 88]:

Given A nonnegative and a positive integer $m \geq 1$, then:

$$(\operatorname{tr} A^2)^m \leq n^{m-1} \operatorname{tr} A^{2m} \quad , \tag{3}$$

$$(\operatorname{tr} A)^2 \leq n \operatorname{tr} A^2 . \tag{4}$$

<u>Proof</u>: The diagonal elements of A^2 are $\sum_k a_{ik} a_{ki}$, $i = 1,2,\cdots,n$. Thus, from the Holder's inequality,

$$(\operatorname{tr} A^2)^m \leq n^{m-1} \sum_i \left(\sum_k a_{ik}a_{ki}\right)^m \quad ,$$

and from 2 (2),

$$(\mathrm{tr}\ A^2)^m \leq n^{m-1}\ \mathrm{tr}\ A^{2m} \quad ,$$

which proves (3). Inequality (4) follows from 1 (4).

□

(g) For singular values:

If A is nonnegative (primitive) then:

$$\mathrm{tr}\ A^2 / n \leq (<)\ \sigma_1{}^2 \quad . \tag{5}$$

Further, in case A is cyclic with period $d > 1$ then:

$$d\ \mathrm{tr}\ A^2 / n \leq \sum_{i=1}^{d} \sigma_i \quad . \tag{6}$$

□

§10:5.4.3

In this section we shall give bounds for the imaginary parts of the eigenvalues of A. Their proofs are given in [64]. They involve the use of the facts that λ_1 is real and that the complex eigenvalues of A occur in conjugate pairs. We shall omit these proofs.

As in Chapter 5, let,

$$K_C{}^u = \begin{cases} (||C||^4 - \frac{1}{8}\,||D||^2)^{1/2} & , \quad \text{if } ||C|| \geq ||B|| \\ ||C||^2 - \frac{1}{12}\,\frac{||D||^2}{||A||^2} & , \quad \text{otherwise,} \end{cases} \tag{1}$$

and

$$K_C{}^\ell = ||C||^2 - \left(\frac{n^3-n}{48}\right)^{1/2} ||D|| \,, \tag{2}$$

where, $B = \frac{1}{2}(A+A^*)$, $C = \frac{1}{2i}(A-A^*)$ and $D = AA^* - A^*A$.

Finally, let the eigenvalues of A be ordered as:

$$\mathrm{Im}(\lambda_1) \geq \mathrm{Im}(\lambda_2) \geq \cdots \geq \mathrm{Im}(\lambda_n) \quad .$$

(c) For λ_k :

For $n \geq 3$ and $K_C{}^u$ and $K_C{}^\ell$ as above,

$$\{\max(0\,, \frac{1}{2p^2}\,K_C{}^\ell)\}^{1/2} \leq \mathrm{Im}(\lambda_1) \leq (\tfrac{1}{2}\,K_C{}^u)^{1/2} \quad ,$$

where $p = [\frac{n-1}{2}]$. Also for $k \leq [\frac{n-1}{2}]$,

$$\mathrm{Im}(\lambda_k) \leq (\tfrac{1}{2k}\,K_C{}^u)^{1/2} \quad .$$

Proof: See [64].

□

(e) For sum of eigenvalues:

Given $1 \le j \le k \le [\frac{n-1}{2}] = p$ and $K_C{}^u$, $K_C{}^\ell$ as in (1) and (2),

$$\frac{1}{k-j+1} \sum_{j}^{k} \mathrm{Im}(\lambda_i) \le (\frac{1}{2k} K_C{}^u)^{1/2} \quad ;$$

$$\{\max (0 , \frac{1}{2p^2} K_C{}^\ell)\}^{1/2} \le \frac{1}{k} \sum_{1}^{k} \mathrm{Im}(\lambda_i) \quad ;$$

and

$$|\mathrm{Im}(\lambda_j) - \mathrm{Im}(\lambda_k)| \le \left[\frac{p\, K_C{}^u - K_C{}^\ell}{2p}\right]^{1/2} (\frac{1}{j} + \frac{1}{p-k+1})^{1/2} \quad .$$

Proof: See [64].

□

§10:6

In this section we shall give results which involve only the row or column sums. We recall that,

$$R_i = \sum_j |a_{ij}| = \sum_j a_{ij} \quad \text{and} \quad C_i = \sum_j |a_{ji}| = \sum_j a_{ji} \quad . \tag{1}$$

The following is a well-known result due to Frobenius:

(a) For $r = \lambda_1 = \max_i |\lambda_i|$:

(i) Given $A \geq 0$,

$$\min_i R_i \leq r \leq \max_i R_i \quad . \tag{2}$$

Further, if A is irreducible then equality on either side implies equality throughout.

Proof: If we prove (2) for a positive matrix then by continuity it will also hold for A nonnegative. So we assume $A > 0$. If x is a positive eigenvector corresponding to r then, $Ax = rx$. Thus,

$$\sum_k a_{ik}x_k = r\, x_i \quad , \quad i = 1,2,\cdots,n \quad ,$$

which gives,

$$r \max_i x_i \leq \sum_k a_{ik} \max_i x_i \quad ,$$

$$r \min_i x_i \geq \sum_k a_{ik} \min_i x_i \quad .$$

Now since $\min_i x_i > 0$, (2) follows. Further, for irreducible

A , let equality hold on one side of (2) only. Then we can increase or decrease the elements of A (keeping A irreducible) so that all row sums become equal and r is unchanged. But from Theorem 0 (14) if B is such that $0 \leq B \leq A$ then, $\rho(B) \leq \rho(A)$ and equality holds only if $B = A$. Thus we have a contradiction, which completes the proof.

□

(ii) Given $A \geq 0$,

$$\min_i \frac{\sum_j a_{ij}x_j}{x_i} \leq r \leq \max_i \frac{\sum_j a_{ij}x_j}{x_i} \quad , \tag{3}$$

where x is any positive vector. In case A is irreducible, equality on either side of (3) implies equality on both sides.

<u>Proof</u>: For given $x = (x_i) > 0$, let $X = \mathrm{diag}(x_1, x_2, \cdots, x_n)$. Then the result follows when (2) is applied to $X^{-1}AX$.

□

(c) For λ_k :

If A is cyclic with period d then:

$$\min_i R_i \leq |\lambda_i| \leq \max_i R_i \quad , \quad i = 1,2,\cdots,d \quad . \tag{4}$$

<u>Proof</u>: The proof is immediate from Theorem 0 (15) and (2).

□

(g) For singular values:

Given $A \geq 0$,

$$\min_i R_i \leq \sigma_1 \quad . \tag{5}$$

<u>Proof</u>: Since $\lambda_1 \leq \sigma_1$, (5) follows from (2).

□

§10:6.2

Here, we shall give bounds which involve only row sums and the diagonal elements of A .

(a) For $r = \lambda_1 = \max_i |\lambda_i|$:

If $A \geq 0$ is irreducible then:

$$\min_{1 \neq j} M(i,j) \leq r \leq \max_{1 \neq j} M(i,j) \quad , \tag{1}$$

where,

$$M(i,j) = \frac{1}{2} \{a_{ii} + a_{jj} + [(a_{ii} - a_{jj})^2 + 4 P_i P_j]^{1/2}\} \quad , \tag{2}$$

and

$$P_i = \sum_{j \neq i} a_{ij} \quad , \quad i,j = 1,2,\cdots,n \quad .$$

Proof: See [8].

□

(g) For singular values:

If $A \geq 0$ is irreducible then,

$$\min_{i \neq j} M(i,j) \leq \sigma_1 \quad , \tag{3}$$

where $M(i,j)$ is given by (2).

Proof: Since $\lambda_1 \leq \sigma_1$, (3) follows from (1).

□

§10:7

Here we shall give results which involve only det A. Their proofs are immediate from the theorems of Section 0.

(a) For $r = \lambda_1 = \max_i |\lambda_i|$:

If A is nonnegative (primitive) then:

$$|\det A|^{1/n} \leq (<) \; r \quad . \tag{1}$$

□

(c) For λ_k :

If A is cyclic with period $d > 1$ then:

$$|\det A|^{1/n} \leq |\lambda_k| \quad , \quad k = 1,2,\cdots,d \quad . \tag{2}$$

□

(g) For singular values:

Given A nonnegative (primitive),

$$|\det A|^{1/n} \leq (<) \; \sigma_1 \quad .$$

Further, if A is cyclic with period $d > 1$ then:

$$d \; |\det A|^{1/n} \leq \sum_{i=1}^{d} \sigma_i \quad .$$

□

§10:7.6

Given the row sums and the $\det A$, we have the following lower bound for the condition number:

(h) For the condition number:

Given A, if $\det A \neq 0$ then:

$$0 \leq \min_i R_i / |\det A|^{1/n} \leq \frac{|\lambda_1|}{|\lambda_n|} \leq c(A) ,$$

where $R_i = \sum_j a_{ij}$.

Proof: The above inequality is immediate from 6 (2) and the fact that $|\lambda_n| \leq |\det A|^{1/n}$. Note that $\det A \neq 0$ implies, $\min_i R_i > 0$. □

§10:8

Given $A \geq 0$ and that the eigenvalues λ_i , $i = 1,2,\cdots,n$ of A are real, let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Since A is nonnegative $\operatorname{tr} A = \sum_i \lambda_i \geq 0$. When the eigenvalues of A are such that $\lambda_1 \geq 0 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then these two facts guarantee that there exists a symmetric matrix S , with λ_i , $i = 1,2,\cdots,n$ as its eigenvalues (see [6, pg. 90]). However, with this information one cannot conclude that A is symmetric. For example,

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} ,$$

is nonnegative, with $\lambda_1 = 3$ and $\lambda_2 = 1$. But A is not symmetric.

□

CHAPTER 11

NORMAL MATRIX

§11:0 Preliminaries.

The matrix A is called normal if $AA^* = A^*A$. Thus, Hermitian, Unitary and Skew - Hermitian matrices are examples of normal matrices. Normal matrices have some special properties. In general a matrix is diagonalizable if all its eigenvalues are distinct. However, a normal matrix is always unitarily diagonalizable. Below we give some properties of a normal matrix.

First we give some conditions which are equivalent to a matrix being normal. Their proofs and several other equivalent conditions can be found in [16].

Theorem 1. Given A, each of the following conditions is equivalent to A being normal:

(i) $AA^* - A^*A$ is positive semidefinite.

(ii) A^* can be represented as a polynomial in A.

(iii) A can be reduced to a diagonal form by a unitary similarity transformation.

(iv) $\sum_i |\lambda_i|^2 = \operatorname{tr} AA^*$.

(v) The singular values of A are $|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|$.

(vi) The eigenvalues of $\frac{1}{2}(A+A^*)$ and $\frac{1}{2i}(A-A^*)$ are precisely $\mathrm{Re}(\lambda_i)$ and $\mathrm{Im}(\lambda_i)$, $i = 1,2,\cdots,n$, respectively.

□

The following results are well-known (e.g. see [31, pg. 64]).

Theorem 2. Given normal A , it is Hermitian if and only if its eigenvalues are real.

Proof: Let the eigenvalues of A be real. Then $A = UDU^*$, for some unitary matrix U and real matrix $D = \mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_n)$ and $A^* = (UDU^*)^* = A$. Thus A is Hermitian. The converse, is trivial.

□

Considering the matrix iA the above Theorem yields the following:

Theorem 3. Given normal A , it is skew-Hermitian if and only if the eigenvalues of A are purely imaginary.

□

Theorem 4. Given normal A , it is unitary if and only if the eigenvalues of A have absolute value one.

Proof: Given $|\lambda_i| = 1$, $i = 1,2,\cdots,n$, Theorem (1) implies $\sigma_i = 1$, $i = 1,2,\cdots,n$. Thus from the Singular value decomposition theorem (see Theorem II:4:0 (4)), we get $A = UV$ for some unitary matrices U and V . Now, as the product of two unitary matrices is a unitary matrix, we conclude that A is unitary. Conversely, A unitary implies $AA^* = I$ and again we have $\sigma_i = 1$, $i = 1,2,\cdots,n$. Finally, from Theorem (1), $\sigma_i = |\lambda_i| = 1$, which completes the proof.

□

The following results are given in [45] and [46]:

Theorem 5. The diagonal elements of a normal matrix A are its eigenvalues if and only if A is a diagonal matrix.

Proof: Let a_{ii}, $i = 1,2,\cdots,n$ be the eigenvalues of A then, from Theorem (1), $|a_{ii}|$, $i = 1,2,\cdots,n$ are the singular values of A. Thus, $\operatorname{tr} AA^* = \sum_{i,j} |a_{ij}|^2 = \sum_i |a_{ii}|^2$, gives $a_{ij} = 0$, $i \neq j$. Conversely if A is diagonal then clearly a_{ii}, $i = 1,2,\cdots,n$ are its eigenvalues. □

Theorem 6. Given normal A,

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n| , \tag{1}$$

if and only if $A = cU$, for some scalar c and unitary U.

Proof: Proof follows at once from the Singular value decomposition theorem (see Theorem II:4:0 (4)).

□

Theorem 7. Given normal A then:

$$\lambda_1 = \lambda_2 = \cdots\cdot = \lambda_n , \tag{2}$$

if and only if A is a scalar matrix.

Proof: From Theorem (1), there exists a unitary U such that $A = UDU^*$, where $D = \operatorname{diag}(\lambda_1,\lambda_2,\cdots,\lambda_n)$. Thus from (2) $A = \lambda_1 UU^* = \lambda_1 I$. The converse is clear.

□

<u>Remark 8.</u> From Theorem (1), we have that for a normal matrix $\sigma_i = |\lambda_i|$, $i = 1,2,\cdots,n$. Thus, bounds for $|\lambda_i|$'s are also valid for σ_i's . In view of this we shall not give bounds for singular values explicitly. We also notice that $Re(\lambda_i)$ and $Im(\lambda_i)$, $i = 1,2,\cdots,n$ are the eigenvalues of Hermitian matrices, $B = \frac{1}{2}(A+A^*)$ and $C = \frac{1}{2i}(A-A^*)$, respectively. Thus, bounds for real and imaginary parts of the eigenvalues A can be derived by considering B and C .

$\square$

§11:2

In this section we shall give results which involve only the diagonal elements of A .

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

Given A ,

$$\max_i |a_{ii}| \leq |\lambda_1| \quad . \tag{1}$$

Equality holds if and only if A is diagonal.

Proof: Since for normal A , $\sigma_1 = |\lambda_1|$, inequality (1) is immediate from Theorem II:4:4 (7). The condition for equality is clear from Theorem (5).

□

(c) For λ_k :

Given A ,

$$\max_i \operatorname{Re}(a_{ii}) \leq \max_i \operatorname{Re}(\lambda_i) \quad ; \tag{2}$$

$$\max_i \operatorname{Im}(a_{ii}) \leq \max_i \operatorname{Im}(\lambda_i) \quad ; \tag{3}$$

$$\min_i \operatorname{Re}(\lambda_i) \leq \min_i \operatorname{Re}(a_{ii}) \quad ; \tag{4}$$

and

$$\min_i \operatorname{Im}(\lambda_i) \leq \min_i \operatorname{Im}(a_{ii}) \quad ; \tag{5}$$

Proof: As mentioned in the Remark (8), for normal A, $\operatorname{Re}(\lambda_i)$ and $\operatorname{Im}(\lambda_i)$, $i = 1,2,\cdots,n$ are the eigenvalues of Hermitian

matrices $B = \frac{1}{2}(A+A^*)$ and $C = \frac{1}{2i}(A-A^*)$ respectively. All of the above inequalities follow from the fact that if P is Hermitian then $\lambda_n \le \min_i p_{ii} \le \max p_{ii} \le \lambda_1$ (see 12:2 (1) and 12:2 (2)).

(d) For the spread:

Given A,

$$3^{1/2} \max_i |a_{ii}| \le 3^{1/2} \max_{i,j} |a_{ij}| \le sp(A) \quad . \tag{6}$$

Proof: See [40].

□

(e) For sum of eigenvalues:

If the diagonal elements of A ordered as $|a_{11}| \ge |a_{22}| \ge \cdots \ge |a_{nn}|$ then:

$$\sum_{i=1}^{k} |a_{ii}| \le \sum_{i=1}^{k} |\lambda_i| \quad , \quad k = 1,2,\cdots,n \quad ; \tag{7}$$

$$\sum_{i=1}^{n-1} |a_{ii}| - |a_{nn}| \le \sum_{i=1}^{n-1} |\lambda_i| - |\lambda_n| \quad . \tag{8}$$

Equality holds in (7) and (8) if and only if A is a diagonal matrix.

Proof: Since A is normal we have $\sigma_i = |\lambda_i|$, $i = 1,2,\cdots,n$. Now, (7) and (8) follow from 2:0 (3) and 2:0 (4), respectively. The condition for equality is clear from Theorem (5).

□

§11:4

Given $\operatorname{tr} AA^*$ for normal A, we have the following result:

(e) For sum of eigenvalues:

Given $\operatorname{tr} AA^*$,

$$\sum_i |\lambda_i|^2 = \operatorname{tr} AA^* \quad . \tag{1}$$

Proof: Since for A normal, $\sigma_i = |\lambda_i|$, $i = 1,2,\cdots,n$, (1) follows.

□

§11:4.3

Given $\operatorname{tr} A^2$, $\operatorname{tr} AA^*$ and that A is normal, we have the following results for the sum of eigenvalues of A :

(e) For sum of eigenvalues:

Given normal A,

$$|\operatorname{tr} A^2| \leq \sum_i |\lambda_i|^2 = \operatorname{tr} AA^* \quad ; \qquad (1)$$

$$\sum_i (\operatorname{Re}(\lambda_i))^2 = \operatorname{tr} B^2 = \frac{1}{2}(\operatorname{tr} AA^* + \operatorname{Re}(\operatorname{tr} A^2)) \quad ; \qquad (2)$$

$$\sum_i (\operatorname{Im}(\lambda_i))^2 = \operatorname{tr} C^2 = \frac{1}{2}(\operatorname{Re}(\operatorname{tr} A^2) - \operatorname{tr} AA^*) \quad . \qquad (3)$$

Proof: Since A is normal, we have $\sigma_i = |\lambda_i|$ and that $\operatorname{Re}(\lambda_i)$ and $\operatorname{Im}(\lambda_i)$ are the eigenvalues of B and C, $i = 1,2,\cdots,n$. Now the proof is clear.

□

§11:4.3.1

Given normal A , $\operatorname{tr} A$, $\operatorname{tr} A^2$ and $\operatorname{tr} AA^*$, we have $\operatorname{tr} B = \operatorname{Re}(\operatorname{tr} A) / n$, $\operatorname{tr} C = \operatorname{Im}(\operatorname{tr} A) / n$ and $\operatorname{tr} B^2$ and $\operatorname{tr} C^2$ as given by 4.3 (2) and 4.3 (3). Using this information several inequalities, for real and imaginary parts of the eigenvalues of A have been derived in [63]. All of the results to follow are proved in [63].

First, we define:

$$m_B = \operatorname{Re}(\operatorname{tr} A) / n = \operatorname{tr} B / n \quad , \quad m_C = \operatorname{Im}(\operatorname{tr} A) / n = \operatorname{tr} C \quad , \tag{1}$$

$$s_B^2 = \operatorname{tr} B^2 / n - m_B^2 \quad \text{and} \quad s_C^2 = \operatorname{tr} C^2 / n - m_C^2 \quad , \tag{2}$$

and let the real and imaginary parts of the eigenvalues of A , arranged in decreasing order be denoted by $\lambda_i^{(B)}$ and $\lambda_j^{(C)}$, respectively. That is, $\lambda_i = \lambda_j^{(B)} + i\,\lambda_k^{(C)}$ for some j and k , not necessarily equal.

(c) For λ_k :

With m_T and s_T as above, $T = B, C$,

$$m_T + s_T / (n-1)^{1/2} \leq \lambda_1^{(T)} \quad , \tag{3}$$

and

$$\lambda_n^{(T)} \leq m_T - s_T / (n-1)^{1/2} \quad . \tag{4}$$

Equality holds in (3) if and only if the $(n-1)$ largest $\lambda_j^{(T)}$'s are equal. Further, equality holds in (4) if and only if the $(n-1)$ smallest $\lambda_j^{(T)}$'s are equal.

Proof: See [63].

□

(e) For sum of eigenvalues:

(i) With m_T and s_T ($T = B,C$) as before, we have:

$$\frac{1}{\ell}\sum_{1}^{\ell} \lambda_i(T) \geq \begin{cases} m_T + \dfrac{s_T}{(n-1)^{1/2}} & \text{if } \ell \leq n/2 \\[2ex] m_T + \dfrac{s_T(n-\ell)}{\ell(n-1)^{1/2}} & \text{if } \ell \geq n/2 \end{cases} \tag{5}$$

Equality holds if and only if

$$\lambda_1^{(T)} = \lambda_2^{(T)} = \cdots = \lambda_{n-1}^{(T)} \qquad \text{when} \quad \ell < n/2 \ ,$$

$$\lambda_1^{(T)} = \lambda_2^{(T)} = \cdots = \lambda_{n-1}^{(T)}$$

$$\text{or} \quad \lambda_2^{(T)} = \lambda_3^{(T)} = \cdots = \lambda_n^{(T)} \qquad \text{when} \quad \ell = n/2 \ ,$$

$$\lambda_2^{(T)} = \lambda_3^{(T)} = \cdots = \lambda_n^{(T)} \qquad \text{when} \quad \ell > n/2 \ .$$

Also,

$$\frac{1}{n-k+1}\sum_{i=k}^{n} \lambda_j^{(T)} \leq \begin{cases} m_T - \dfrac{s_T(k-1)}{(n-k+1)(n-1)^{1/2}} & \text{if } k \leq \frac{n}{2}+1 \\[2ex] m_T - \dfrac{s_T}{(n-1)^{1/2}} & \text{if } k \geq \frac{n}{2}+1 \ . \end{cases} \tag{6}$$

Equality holds if and only if

$$\lambda_1^{(T)} = \lambda_2^{(T)} = \cdots = \lambda_{n-1}^{(T)} \qquad \text{when} \quad k < \frac{n}{2} + 1 \ ,$$

$$\lambda_1^{(T)} = \lambda_2^{(T)} = \cdots = \lambda_{n-1}^{(T)} \qquad \text{or}$$

$$\lambda_2^{(T)} = \lambda_3^{(T)} = \cdots = \lambda_n^{(T)} \quad \text{when} \quad k = \frac{n}{2} + 1 ,$$

$$\lambda_2^{(T)} = \lambda_3^{(T)} = \cdots = \lambda_n^{(T)} \quad \text{when} \quad k > \frac{n}{2} + 1 .$$

<u>Proof</u>: See [63].

□

(iii) With s_T , $T = B,C$ as in (2),

$$2\, s_T \leq \lambda_1^{(T)} - \lambda_n^{(T)} . \tag{7}$$

In case n is even, say $n = 2q$, equality holds if and only if

$$\lambda_2^{(T)} = \lambda_3^{(T)} = \cdots = \lambda_q^{(T)} \text{ and } \lambda_{q+1}^{(T)} = \lambda_{q+2}^{(T)} = \cdots = \lambda_n^{(T)} . \tag{8}$$

Furthermore, if n is odd, say $n = 2q \pm 1$, then:

$$2\, s_T \;\; n/(n^2-1)^{1/2} \leq \lambda_1^{(T)} - \lambda_n^{(T)} . \tag{9}$$

Equality holds if and only if (8) holds.

<u>Proof</u>: See [63].

□

§11:6

In this section we shall give results which involve only the row sums. We recall that,

$$R_i = \sum_j |a_{ij}| \quad , \qquad C_i = \sum_j |a_{ji}| \quad , \tag{1}$$

$$R = \max_i R_i \quad \text{and} \quad C = \max_i C_i \quad . \tag{2}$$

(a) For $|\lambda_1| = \max_i |\lambda_i|$:

Given A ,

$$\max\left(\frac{R}{n^{1/2}} , \max_{s,t} \frac{1}{(st)^{1/2}} \left|\sum_{j=1}^{t} \sum_{i=1}^{s} a_{ij}\right|\right) \leq |\lambda_i| \leq R \quad . \tag{3}$$

<u>Proof</u>: Since $|\lambda_1| = \sigma_1$, the inequality on the left follows from 6:0 (16) and Theorem II:4:4 (1). The inequality on the right is 6:0 (7).

□

(b) For $|\lambda_n| = \min_i |\lambda_i|$:

Given A ,

$$|\lambda_n| \leq \min_i R_i \quad . \tag{4}$$

<u>Proof</u>: Since A is normal, $\sigma_n = |\lambda_n|$. Now (4) follows from II:4:4 (10).

□

(d) For the spread:

If

$$S_i = \sum_j a_{ij} \quad , \quad i = 1,2,\cdots,n \quad , \tag{5}$$

is real then:

$$3^{1/2}\, \nu \leq sp(A) \quad , \tag{6}$$

where,

$$\nu^2 = \frac{1}{n}\sum_i S_i^{\,2} - m^2 \quad , \quad m = \frac{1}{n}\sum_i S_i \quad . \tag{7}$$

Further, if S_i , $i = 1,2,\cdots,n$ are ordered so that

$$S_{i_1} \geq S_{i_2} \geq \cdots \geq S_{i_n} \quad ,$$

then:

$$\left\{ \frac{3}{2n} \sum_{j=1}^{[\frac{n+1}{2}]} (S_{i_j} - S_{i_{n-j+1}})^2 \right\}^{1/2} \leq sp(A) \quad . \tag{8}$$

<u>Proof</u>. See [23].

□

(e) For sum of eigenvalues:

With R_i , $i = 1,2,\cdots,n$ as in (1),

$$\sum_i R_i / n \leq \sum_i |\lambda_i| \leq \sum_i R_i \quad ; \tag{9}$$

$$\sum_i R_i^{\,2} / n \leq \sum_i |\lambda_i|^2 \leq \sum_i R_i^{\,2} \quad . \tag{10}$$

<u>Proof</u>: As A is normal the above results follow from 6:0 (19) and 6:0 (20). $\square$

§11:7

Given normal A, clearly results of Chapter 7 hold. Further, the equality conditions given in Chapter 7 become applicable. In addition we have the following result for the condition number:

(h) For the condition number:

If $\det A \neq 0$ then:

$$1 + \frac{2\, s_A}{(\operatorname{tr} A^* A/n)^{1/2}} \leq \frac{|\lambda_1|}{|\lambda_n|} = c(A) \quad , \tag{1}$$

where,

$$m = \operatorname{tr} A / n \quad \text{and} \quad s_A^2 = \operatorname{tr} AA^* / n - |m|^2 \quad .$$

Furthermore, if

$$|\operatorname{tr} A|^2 > (n-1) \operatorname{tr} AA^* \quad ,$$

then

$$c(A) = \frac{|\lambda_1|}{|\lambda_n|} \leq 1 + \frac{(2n)^{1/2}\, s_A}{|m| - s_A (n-1)^{1/2}} \quad . \tag{2}$$

<u>Proof</u>: See [63].

□

§11:8

Given that a normal matrix A has real eigenvalues then A is necessarily Hermitian (see Theorem 0 (2)).

$\square$

§11:9.6.1

Given $R_i = \sum_j a_{ij}$, $i = 1,2,\cdots,n$ and $\mathrm{tr}\, A$, we have the following bound for the spread:

(d) For the spread:

Given A ,

$$\frac{1}{n-1}\left|\sum_i R_i - \mathrm{tr}\, A\right| \le \mathrm{sp}(A) \quad .$$

<u>Proof:</u> See [23]. □

CHAPTER 12

HERMITIAN MATRIX

§12:0 Preliminaries.

The matrix A is called Hermitian if $A^* = A$. In case a Hermitian matrix is real, it is known as a symmetric matrix. All the eigen values of a Hermitian matrix A are real. For, if $Ax = \lambda x$ then,

$$\lambda(x,x) = (Ax,x) = (x,Ax) = \bar{\lambda}(x,x) \quad .$$

Throughout this chapter, we shall assume that the eigenvalues of A are ordered as:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad . \tag{1}$$

Given A Hermitian, it is clearly normal. Thus, for A Hermitian, all the results of Chapters 8 and 11 hold. In particular, we have the following:

Theorem 1: If A is Hermitian, then it has a set of orthonormal eigen-vectors, i.e. $A = UDU^*$, where U is unitary and $D = \text{diag}(\lambda_1,\lambda_2,\cdots,\lambda_n)$. □

Theorem 2: Given A Hermitian, we have

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n \tag{2}$$

if and only if A is a real scalar matrix. □

In addition, a Hermitian matrix has other very useful properties. First, we shall define the Rayleigh quotient:

Defintion 3. Given A Hermitian and a vector $x \neq 0$ then

$$\rho(x) = \frac{(x,Ax)}{(x,x)} , \tag{3}$$

is called the Rayleigh quotient. □

Since for any $x \in ₵^n$, $\overline{(x,Ax)} = (Ax,x) = (x,A^*x) = (x,Ax)$, we conclude that $\rho(x)$ is always real.

Now, we shall prove the well-known Rayleigh's principle relating the eigenvalues of A and the Rayleigh quotient.

Theorem 4. If the Hermitian matrix A has eigenvalues,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n ,$$

then, for any $x \in ₵^n$,

$$\lambda_n \leq \rho(x) \leq \lambda_1 , \tag{4}$$

and

$$\lambda_1 = \max_{x\neq 0} \rho(x) = \max_{x\neq 0} \frac{(x,Ax)}{(x,x)} , \quad \lambda_n = \min_{x\neq 0} \rho(x) = \min_{x\neq 0} \frac{(x,Ax)}{(x,x)} . \tag{5}$$

Proof: Let x_i , $i = 1,2,\cdots,n$ be a set of orthonormal eigenvectors of A , such that $Ax_i = \lambda_i x_i$. Then, for any x , $x = \sum_i \alpha_i x_i$ for some scalars α_i , $i = 1,2,\cdots,n$, and moreover, we have,

$$Ax = \sum_i \alpha_i \, Ax_i = \sum_i \alpha_i \lambda_i x_i \quad ,$$

$$\rho(x) = \frac{(x,Ax)}{(x,x)} = \frac{\sum_i |\alpha_i|^2 \lambda_i}{\sum_i |\alpha_i|^2} \quad . \tag{6}$$

Clearly, $\rho(x) \le \lambda_1$ and $\rho(x) \ge \lambda_n$. Finally, (5) follows noticing that. $\rho(x_1) = \lambda_1$ and $\rho(x_n) = \lambda_n$. □

More generally, we have the following:

Theorem 5. Let A be a Hermitian matrix with eigenvalues, $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ and corresponding orthonormal eigenvectors $x_1, x_2, \cdots, x_n$. Then:

$$\lambda_k = \max_{\substack{(x,x_i)=0 \\ i=1,2,\cdots,k-1}} \rho(x) \quad . \tag{7}$$

□

Next, we state the well-known Courant-Fisher Theorem (e.g. see [42, pg. 414]):

Theorem 6: (Courant-Fisher Theorem). Given Hermitian A ,

$$\lambda_1 = \max_{x \neq 0} \rho(x) \quad ,$$

$$\lambda_k = \max_{\substack{||y_i||=1 \\ i=1,2,\cdots,k-1}} \; \min_{(x,y_i)=0} \rho(x) \;, \; k = 2,3,\cdots,n \quad . \tag{8}$$

$$\lambda_n = \min_{x \neq 0} \rho(x) \quad ,$$

$$\lambda_{n-k+1} = \max_{\substack{||y_i||=1 \\ i=1,2,\cdots,k-1}} \min_{(x,y_i)=0} \rho(x) \ , \ k=2,3,\cdots,n \ . \tag{9}$$

□

Finally, we give the interlacing theorem for a Hermitian matrix:

Theorem 7: Given Hermitian A, let A_i be its principal submatrix obtained by deleting the ith row and column and $\lambda_1(A_i) \geq \lambda_2(A_i) \geq \cdots \geq \lambda_{n-1}(A_i)$, be the eigenvalues of A_i. Then:

$$\lambda_{k+1} \leq \lambda_k(A_i) \leq \lambda_k \ , \quad k=1,2,\cdots,n-1 \ . \tag{10}$$

Proof: The principal submatrix A_i is Hermitian of order $(n-1)$. Thus, from (7) we have,

$$\lambda_k(A_i) = \max_{x\neq 0} \frac{(x,A_i x)}{(x,x)} \ , \tag{11}$$

where x is a $(n-1)$ vector. Now, if $y = (y_j)$ is such that $y_j = x_j$, $j < i$, $y_i = 0$ and $y_{j+1} = x_j$, $j \geq i$, then from (7),

$$\lambda_k \geq \max_y \frac{(y,Ay)}{(y,y)} = \lambda_k(A_i) \ ,$$

which establishes the inequality on the right in (10).

To obtain the inequality on the left in (10), we shall employ the Courant-Fisher Theorem: From (9), we have

$$\lambda_k(A_i) = \max_{\substack{||y_i||=1 \\ i=1,2,\cdots,n-1-k}} \min_{(x,y_i)=0} \frac{(x,A_i x)}{(x,x)} \ , \tag{12}$$

where, x and y_i are $(n-1)$ vectors. Also,

$$\lambda_{k+1} = \max_{\substack{||v_i||=1 \\ i=1,2,\cdots,n-1-k}} \min_{(u,v_i)=0} \frac{(u,Au)}{(v,Av)} . \tag{13}$$

Now, considering vectors u, with zero ith component, in (13), we essentially maximize over vectors v_j whose ith components are zero. Thus, (10) follows. □

<u>Remark 8</u>. Since a Hermitian matrix A is normal, we have,

$$\sigma_i = |\lambda_i| \quad , \quad i = 1,2,\cdots,n \quad .$$

Therefore, we shall not mention results for the singular values explicitly. Also we note that for Hermitian A,

$$\sum_i \lambda_i^2 = \sum_i \sigma_i^2 = \text{tr } A^2 \quad .$$ □

§12:2

In this section we shall give results which involve only the diagonal elements of a Hermitian matrix. We note that the diagonal elements of a Hermitian matrix are always real.

The inequalities (1) and (2) below, are immediate from Theorem (4).

(a) For $\lambda_1 = \max_i \lambda_i$:

Let A be a Hermitian matrix. Then

$$\max_i a_{ii} \leq \lambda_1 \quad . \tag{1}$$

□

(b) For $\lambda_n = \min_i \lambda_i$:

Given Hermitian A ,

$$\lambda_n \leq \min_i a_{ii} \quad . \tag{2}$$

□

(d) For the spread:

Given A Hermitian,

$$2 \max_i |a_{ii}| \leq 2 \max_{i,j} |a_{ij}| \leq sp(A) \quad . \tag{3}$$

Proof: See [41]. □

Recall, that if $x = (x_i)$ and $y = (y_i)$ are any two real vectors then y is said to majorize x if

$$\sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)} \quad , \quad k = 1,2,\cdots,n \quad , \tag{4}$$

and equality holds for $k = n$, where $x_{(i)}$, $y_{(i)}$ are components of the vectors x and y arranged in decreasing order. Further, if y majorizes x, we write

$$x \prec y \quad . \tag{5}$$

(e) For sum of eigenvalues:

The following result is given in [32, pg. 218]:

Given Hermitian A, let a and λ be two vectors such that $a = (a_{ii})$ and $\lambda = (\lambda_i)$. Then:

$$a \prec \lambda \quad . \tag{6}$$

<u>Proof</u>: Without loss of generality, we let $a_{11} \geq a_{22} \geq \cdots \geq a_{nn}$ and for $1 \leq k \leq n$ define $A_k = (a_{ij})$, $i,j = 1,2,\cdots,k$. Let the eigenvalues of A_k be $\lambda_1(A_k) \geq \lambda_2(A_k) \geq \cdots \geq \lambda_k(A_k)$. Then from Theorem 0(7), for $1 \leq k \leq n-1$,

$$\lambda_1(A_{k+1}) \geq \lambda_1(A_k) \geq \lambda_2(A_{k+1}) \geq \cdots \geq \lambda_k(A_k) \geq \lambda_{k+1}(A_{k+1}) \quad .$$

Thus, by definition of trace,

$$\sum_{i=1}^{k} a_{ii} = \sum_{i=1}^{k} \lambda_i(A_k) \leq \sum_{i=1}^{k} \lambda_i(A_{k+1})$$
$$\vdots$$
$$\leq \sum_{1}^{k} \lambda_i \quad .$$

Finally, since $\operatorname{tr} A = \sum_i a_{ii} = \sum_i \lambda_i$, (6) follows. □

§12:3.1

Given $\operatorname{tr} A$ and $\operatorname{tr} A^2$ the results of §8:3.1 hold. In addition, we have the following:

(c) For λ_k :

If for Hermitian A , $\operatorname{tr} A \geq 0$ and

$$\operatorname{tr}^2 A \geq (>)(n-1) \operatorname{tr} A^2 \quad , \tag{1}$$

then A is positive semidefinite (positive definite). That is,

$$\lambda_k \geq (>)\ 0 \quad , \quad k = 1,2,\cdots,n \ .$$

<u>Proof</u>: It is shown in [63] that

$$\lambda_n \geq \frac{\operatorname{tr} A}{n} - \left(\frac{\operatorname{tr} A^2}{n} - \frac{\operatorname{tr}^2 A}{n^2}\right)^{1/2} (n-1)^{1/2} \quad ,$$

which is nonnegative (positive) if (1) holds. Thus the result follows. □

(h) For the condition number:

If for Hermitian A , $\operatorname{tr} A > 0$ and $p = \dfrac{(\operatorname{tr} A)^2}{\operatorname{tr} A^2} - (n-1) > 0$,

then A is positive definite and

$$1 + \frac{2s}{m - s/(n-1)^{1/2}} \leq c(A) = \frac{|\lambda_1|}{|\lambda_n|} \leq \frac{1 + (1-p^2)^{1/2}}{p} \quad , \tag{2}$$

where, $m = \operatorname{tr} A/n$ and $s^2 = \operatorname{tr} A^2 / n - m^2$. When $n > 2$, equality holds on the right if and only if,

$$\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} = \frac{\lambda_1^{\;2} + \lambda_n^{\;2}}{\lambda_1 + \lambda_n} ,$$

and then

$$\frac{\text{tr } A^2}{\text{tr } A} = \frac{\lambda_1^{\;2} + \lambda_n^{\;2}}{\lambda_1 + \lambda_n} .$$

For $n > 2$, equality holds on the left if and only if n is even and A is a scalar matrix.

<u>Proof</u>: See [63]. □

§12:6

Given the row sums of a Hermitian matrix we have the following results:

(a) For $\lambda_1 = \max_i \lambda_i$:

Given Hermitian A ,

$$\sum_i R_i / n \leq \lambda_1 \quad . \tag{1}$$

Proof: Inequality (1) is immediate from 0(5) when $x = (1,1,\cdots,1)'$. It is shown in [36] that (1) is a particularly good estimate for a nonnegative, irreducible symmetric matrix. □

(d) For the spread:

Given A Hermitian, let

$$S_i = \sum_j a_{ij} \quad , \quad i = 1,2,\cdots,n \quad .$$

If S_i , $i = 1,2,\cdots,n$ are real then:

$$2v \leq sp(A) \quad , \tag{2}$$

where,

$$m = \sum_i S_i / n \quad \text{and} \quad v^2 = \frac{1}{n} \sum_i S_i{}^2 - m^2 \quad .$$

Further, if S_i , $i = 1,2,\cdots,n$ are ordered

$$S_{i_1} \geq S_{i_2} \geq \cdots \geq S_{i_n} \quad ,$$

then:

$$\left\{\frac{2}{n}\sum_{j=1}^{[\frac{n+1}{2}]}(S_{i_j} - S_{i_{n-j+1}})^2\right\}^{1/2} \leq sp(A) \quad . \tag{3}$$

Proof: See [23]. □

§12:6.2

The result below is given in [26, pg. 226]. It provides a criteria for checking if a Hermitian matrix is positive definite.

(c) For λ_k :

If the diagonal elements of a Hermitian matrix are all positive and A is diagonally dominant, i.e.

$$a_{ii} > \sum_{j \neq 1} |a_{ij}| = P_i \quad , \quad i = 1,2,\cdots,n \tag{1}$$

then, A is positive definite, that is $\lambda_k > 0$, $k = 1,2,\cdots,n$.

<u>Proof</u>: The diagonal elements of a positive definite matrix must be positive, for $0 < \lambda_n \leq \min_i a_{ii}$. Further, from the Gerschgorin's Theorem (see §6:2) any eigenvalue λ of A satisfies $a_{ii} - \lambda \leq P_i = \sum_{j \neq i} |a_{ij}|$, for some $1 \leq i \leq n$. Thus, from (1) each $\lambda > 0$. Hence the proof. $\square$

§12:10.2

Given a nonnegative symmetric matrix, then we have the following result for the sum of eigenvalues:

(e) For sum of eigenvalues:

If the diagonal elements a_{ii}, and the eigenvalues λ_i, $i = 1,2,\cdots,n$ of a nonnegative symmetric A are ordered as,

$$a_1 \geq a_2 \geq \cdots \geq a_n ,$$

and

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n ,$$

then:

$$\sum_{i=1}^{s} \lambda_i + \lambda_k \geq \sum_{i=1}^{s-1} a_i + a_{k-1} + a_k , \quad 1 \leq s < k < n . \tag{1}$$

<u>Proof</u>: See [6, pg. 97]. □

CHAPTER 13

POSITIVE DEFINITE (SEMIDEFINITE) MATRIX

§13:0 Preliminaries.

Given an $n \times n$ matrix A, it is called positive definite, if A is Hermitian and

$$(x,Ax) > 0 \quad , \text{ for all } \quad 0 \neq x \in \mathbb{C}^n \quad . \tag{1}$$

In case,

$$(x,Ax) \geq 0 \quad , \text{ for all } \quad x \in \mathbb{C}^n \quad ,$$

we say that A is positive semidefinite. Below we give some necessary and sufficient conditions for a matrix to be positive definite (semidefinite).

Theorem 1: Given an $n \times n$ matrix A, the following are equivalent:

(i) A is positive definite (semidefinite);

(ii) A is Hermitian and all its eigenvalues are positive (nonnegative);

(iii) A is Hermitian and all its leading principal minors are positive (all principal minors are nonnegative).

Proof: Let A be positive definite (semidefinite). If λ is an eigenvalue of A with corresponding eigenvector x then we have $(x,Ax) = \lambda(x,x) > (\geq) \ 0$, which is so only if $\lambda > (\geq) \ 0$. Thus (i) implies (ii).

Conversely let all the eigenvalues of A be positive. Then from 12:0 (5) we have $\lambda_n = \min_{x \neq 0} \frac{(x,Ax)}{(x,x)}$, where λ_n is the smallest eigenvalue of A. Therefore, $(x,Ax) > (\geq)\ 0$. Further, if D is any Hermitian matrix then so is any principal submatrix of D. Also, from the interlacing theorem (see Theorem 12:0 (7)) if all the eigenvalues of D are positive (nonnegative) then so are the eigenvalues of any principal submatrix of D. From this we get, (i) implies (iii). Conversely, if all the leading principal minors of A are positive then by induction one can show that A is positive definite (e.g. see [39, pg. 401]). For the positive semidefinite case, see [39, pg. 405]. $\square$

Next, we give some well-known properties of a positive definite (semidefinite) matrix.

<u>Theorem 2</u>. Given a positive definite (semidefinite) matrix A, then:

(i) $a_{ii} > (\geq)\ 0$, $i = 1,2,\cdots,n$. (2)

(ii) Further, if A is positive semidefinite and a_{kk} is zero for some $1 \leq k \leq n$, then each element in the kth row and kth column of A is zero;

(iii) $a_{ii}\, a_{jj} > (\geq)\ |a_{ij}|^2$, $i \neq j$, $1 \leq i$, $j \leq n$: (3)

(iv) there exists k, $1 \leq k \leq n$ such that,

$$|a_{kk}| \geq |a_{ij}| \quad , \quad 1 \leq i \ , \ j \leq n \ ; \tag{4}$$

(v) $\det A > (\geq)\ 0$; (5)

and,

(vi) if $\operatorname{tr} A = 0$ then $A = 0$.

Proof: For positive semidefinite A , inequalities (2) and (3) are clear from the previous theorem. If $a_{kk} = 0$ for some $1 \le k \le n$, it follows from (3) that each element in the kth row and kth column of A is zero. Further, if A is positive definite, then each principal minor of A is positive (see e.g. [31 , pg. 70]). Thus (2) and (3) follow. The inequality (4) is clear from (3); and (5) follows from Theorem (1). Finally, if $\operatorname{tr} A = \sum_i a_{ii} = 0$, then using inequality (4), we get $A = 0$. □

Since the eigenvalues of a positive definite (semidefinite) matrix are positive (nonnegative) and therefore real, we shall always assume that they are ordered as:

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \quad . \tag{6}$$

Remark 3. Given a positive semidefinite matrix A , the singular values equal the eigenvalues, i.e.,

$$\sigma_i = \lambda_i \ , \ i = 1,2,\cdots,n \quad . \tag{7}$$

Also, if A is positive definite,

$$c(A) = \lambda_1 / \lambda_n \quad . \tag{8}$$

□

Finally, we present the Kantorovich inequality:

Theorem 4. (Kantorovich Inequality). If A is positive definite then:

$$1 \leq (x,Ax)(x,A^{-1}x) \leq \frac{1}{4}\,[(\lambda_1/\lambda_n)^{1/2} + (\lambda_n/\lambda_1)^{1/2}]^2 \quad , \qquad (9)$$

for all x such that $||x|| = 1$.

Proof: See [31, pg. 117]. An alternate proof involving Lagrange multipliers is given in [62]. □

§13:1

Given the trace of a positive semidefinite matrix A, we have the following result:

(c) For λ_k:

Given A positive semidefinite,

$$\lambda_k \leq \operatorname{tr} A / k \quad , \quad k = 1,2,\cdots,n \quad . \tag{1}$$

Equality holds if and only if

$$\lambda_1 = \lambda_2 = \cdots = \lambda_k \text{ and } \lambda_{k+1} = \cdots = \lambda_n = 0 \quad .$$

Proof: Since A is positive semidefinite, we have $\lambda_i \geq 0$, $i = 1,2,\cdots,n$. Thus, $\operatorname{tr} A = \sum \lambda_i \geq k\,\lambda_k$, from which (1) follows. The conditions for equality are clear. □

§13:2

Given the diagonal elements of a positive semidefinite matrix, we have the following:

(a) For $\lambda_1 = \max_i \lambda_i$:

Given A ,

$$\max_i a_{ii} \leq \lambda_1 \leq \max_i a_{ii}^{1/2} \sum_j a_{jj}^{1/2} \quad . \tag{1}$$

Proof: From 13:0 (3) we have $\sum_j |a_{ij}| \leq a_{ii}^{1/2} \sum_j a_{jj}^{1/2}$. Now the inequality on the right follows from 6:0 (7). The inequality on the left is 12:2 (1). □

(f) For product of eigenvalues:

Let the diagonal elements of a positive semidefinite matrix be,

$$a_{11} \geq a_{22} \geq \cdots \geq a_{nn} \quad .$$

Then:

$$\prod_k^n \lambda_i \leq \prod_k^n a_{ii} \quad , \quad k = 1,2,\cdots,n \quad . \tag{1}$$

In particular, we have Hadamard's inequality,

$$\det A = \prod_i \lambda_i \leq \prod_i a_{ii} \quad . \tag{2}$$

Equality holds in (2) if and only if A is a diagonal matrix or A has a zero row and column.

Proof: See [32, pg. 223]. □

(h) For the condition number:

For a positive definite matrix A ,

$$1 \le \frac{\max_i a_{ii}}{\min_i a_{ii}} \le \frac{\lambda_1}{\lambda_n} = c(A) \quad . \tag{3}$$

<u>Proof</u>: Inequality (3) is immediate from 12:2 (1) and 12:2 (2), since $\lambda_n > 0$. □

§13:3.1

Given positive definite A , $\operatorname{tr} A$ and $\operatorname{tr} A^2$, we have the following bounds for the condition number.

(h) For the condition number:

Given positive definite A , let $m = \operatorname{tr} A / n$ and $s^2 = \operatorname{tr} A^2 / n - m^2$. If n is even, then:

$$1 + \frac{2s}{m - s(n-1)^{-1/2}} \leq \frac{\lambda_1}{\lambda_n} = c(A) \quad . \tag{1}$$

When $n > 2$ equality holds if and only if A is a scalar matrix. Further, if n is odd then (1) holds, but moreover,

$$1 + \frac{2sn/(n^2-1)^{1/2}}{m - s/(n-1)^{1/2}} \leq \frac{\lambda_1}{\lambda_n} = c(A) \quad . \tag{2}$$

When $n = 3$ equality holds if and only if the two smallest eigenvalues are equal. For $n > 3$ equality holds if and only if A is a scalar matrix.

<u>Proof</u>: See [63]. □

§13:3.2

Given $\operatorname{tr} A^2$ and the diagonal elements of a positive definite matrix, we have the following bounds for $\det A$:

(f) For product of eigenvalues:

(i) Given positive semidefinite A ,

$$\prod_i a_{ii} - \frac{(\operatorname{tr} A)^{n-2}}{n-2} \cdot \frac{\operatorname{tr} A^2 - \sum_i a_{ii}^2}{2} \leq \det A . \tag{1}$$

Proof: See [32, pg. 224]. □

(ii) Given positive definite A ,

$$\det A \leq \prod_i a_{ii} - \lambda_n^{n-1}(\operatorname{tr} A^2 - \sum_i a_{ii}^2)$$

$$\leq \prod_i a_{ii} - (\operatorname{tr} A / n)^{n-1}(\operatorname{tr} A^2 - \sum_i a_{ii}^2) .$$

Proof: The inequality on the left is given in [30], and the inequality on the right follows using $\lambda_n \leq \operatorname{tr} A / n$. □

For the next result we need the following notation:

Let the diagonal elements of A be ordered as,

$$a_1 \geq a_2 \geq \cdots \geq a_n ,$$

$$A_k = \sum_{i=1}^{k} a_i , \quad L_k = \operatorname{tr} A^2 - \sum_{i=k+1}^{n} a_i^2 , \tag{2}$$

$$m_k = A_k/k \quad , \quad s_k^2 = L_k/k - m_k^2 \quad , \tag{3}$$

$$A_{-k} = \sum_{i=n-k+1}^{n} a_i \quad , \quad L_{-k} = \sum_{i=1}^{n-k} a_i^2 \quad , \tag{4}$$

$$m_{-k} = A_{-k}/k \quad \text{and} \quad s_{-k}^2 = L_{-k}/k - m_{-k}^2 \quad . \tag{5}$$

(ii) Given positive definite A , let $\infty = a_0 \geq a_1 \geq \cdots \geq a_n \geq a_{n+1} = -\infty$ contain the ordered diagonal of A . Further, let

$$p_k = m_k - s_k/(k-1)^{1/2} \quad , \tag{6}$$

$$u_k = m_k + s_k(k-1)^{1/2} \quad , \tag{7}$$

$$q_{-j} = m_{-j} + s_{-j}/(j-1)^{1/2} \quad , \tag{8}$$

and

$$b_{-j} = m_{-j} - s_{-j}/(j-1)^{1/2} \quad . \tag{9}$$

Then there exist integers t and r such that $p_k > a_k$, $k = n, n-1, \cdots, t+1$, $a_{t+1} \leq p_t \leq a_t$, $q_{-j} < a_{n-j+1}$, $j = n, n-1, \cdots, r+1$, $a_{n-r} \geq q_{-r} \geq a_{n-r+1}$, and we get

$$\begin{aligned} \det A &\leq u_t(p_t)^{t-1} a_{t+1} \cdots a_n \\ &\leq u_{t+1}(p_{t+1})^{t} a_{t+1} \cdots a_n \\ &\cdots \\ &\leq u_n(p_n)^{n-1} \quad . \end{aligned} \tag{10}$$

Equality holds throughout the first j inequalities if $\lambda_2 = \cdots = \lambda_t = \cdots = \lambda_{t+j-1}$ and $\lambda_{t+j+i} = a_{t+j+i}$, $i = 0, 1, \cdots, n-t-j$, in this case $\lambda_2 = p_{t+j-1}$ and $\lambda_1 = u_{t+j-1}$. Further

$$\det A \geq b_{-r}(q_{-r})^{r-1}\, a_1 \cdots a_{n-r}$$
$$\geq b_{-(r+1)}(q_{-(r+1)})^{r}\, a_1 \cdots a_{n-r-1}$$
$$\cdots$$
$$\geq b_{-n}(q_{-n})^{n-1} \quad . \tag{11}$$

Equality holds throughout the first j inequalities if and only if $\lambda_{n-1} = \cdots = \lambda_{n-r} = \cdots = \lambda_{n-r-j+1}$ and $\lambda_i = a_i$, $i = 1,\cdots,n-r-j$, in this case $\lambda_{n-1} = q_{-(r+j)}$ and $\lambda_n = q_{-(r+j)}$.

<u>Proof</u>: See [17]. □

A procedure for calculating the above bounds for $\det A$ is also given in [17].

§13:6

In this section we shall give results which involve row sums of a positive definite (semidefinite) matrix.

(c) For λ_k :

Given A positive semidefinite, let $R_i = \sum_j |a_{ij}|$ be arranged as $R_{i_1} \geq R_{i_2} \geq \cdots \geq R_{i_n}$. Then:

$$\lambda_k \leq R_{i_k} - \sum_{j=1}^{k-1} \min_{\ell} |a_{i_j \ell}| \leq R_{ik} \quad , \quad k = 1,2,\cdots,n \quad . \tag{11}$$

<u>Proof</u>: For $k = 1$ the result is known (see 6:0 (7)). Let $A_{i_1 i_2 \cdots i_k}$ denote the principal submatrix obtained by deleting the rows and columns $i_1, i_2, \cdots, i_k$. Then by repeated application of 12:0 (10) we have,

$$\lambda_{k+1} \leq \lambda_1 \, (A_{i_1 i_2 \cdots i_k}) \quad , \quad k = 1,2,\cdots,n-1 \quad .$$

Now from 6:0 (7),

$$\lambda_1 (A_{i_1 i_2 \cdots i_k}) \leq R^1_{i_1 i_2 \cdots i_k} \quad ,$$

where $R^j_{i_1 i_2 \cdots i_k}$, $j = 1,2,\cdots,n-k$, are the row sums of $A_{i_1 i_2 \cdots i_k}$ arranged in decreasing order. However,

$$\begin{aligned} R^1_{i_1 i_2 \cdots i_k} &\leq R_{i_{k+1}} - \sum_{j=1}^{k} \min_{\ell} |a_{\ell i_j}| \\ &= R_{i_{k+1}} - \sum_{j=1}^{k} \min_{\ell} |a_{i_j \ell}| \quad , \end{aligned}$$

which completes the proof. □

§13:6.2

Given the row sums and the diagonal elements, of a positive definite matrix, we have the following result for the condition number:

(h) For the condition number:

Given A positive definite,

$$\frac{\max_i R_i}{n^{1/2} \min_i a_{ii}} \leq \frac{\lambda_1}{\lambda_n} = c(A) \quad , \tag{1}$$

where $R_i = \sum_j |a_{ij}|$.

<u>Proof</u>: Since $\sigma_1 = \lambda_1$, from 6:0 (16) we have $\max R_i / n^{1/2} \leq \lambda_1$. Also, from 12:2 (2), $0 < \lambda_n \leq \min_i a_{ii}$. Now (1) follows.

□

§13:7.1

In this section we shall give results which involve the determinant and the trace of a positive definite (semidefinite) matrix.

(a) For $\lambda_1 = \max_i \lambda_i$:

Given positive semidefinite matrix $A \neq 0$,

$$(n/\operatorname{tr} A)^{n-1} \det A \leq \left(\frac{n-1}{\operatorname{tr} A - \lambda_1}\right)^{n-1} \det A \leq \lambda_1 \quad ; \tag{1}$$

and

$$\begin{aligned} \lambda_1 &\leq \operatorname{tr} A - (n-1)(\det A/\lambda_1)^{1/n-1} \\ &\leq \operatorname{tr} A - (n-1)(\det A/\operatorname{tr} A)^{1/n-1} \quad . \end{aligned} \tag{2}$$

Proof: Applying the arithmetic-geometric mean inequality to nonnegative eigenvalues $\lambda_2, \lambda_3, \cdots, \lambda_n$, we have,

$$\det A / \lambda_1 \leq \left(\frac{\operatorname{tr} A - \lambda_1}{n-1}\right)^{n-1} \quad , \tag{3}$$

from which the inequalities on the right of (1) and left of (2) follow. They are also proved in [7, pg. 69]. The inequalities on the left of (1) and right of (2), now follow, since $\operatorname{tr} A / n \leq \lambda_1 \leq \operatorname{tr} A$. □

The following result for λ_n can be proved similarly:

(b) For λ_n :

Given a positive definite matrix A ,

$$\left(\frac{n-1}{\operatorname{tr} A}\right)^{n-1} \det A \leq \left(\frac{n-1}{\operatorname{tr} A - \lambda_n}\right)^{n-1} \det A \leq \lambda_n \quad ; \tag{4}$$

and

$$\lambda_n \leq \operatorname{tr} A - (n-1)\left(\frac{\det A}{\lambda_n}\right)^{1/n-1} \leq \operatorname{tr} A - (n-1)\left(\frac{n \det A}{\operatorname{tr} A}\right)^{1/n-1} \quad . \tag{5}$$

□

The results below, follow from (1), (2), (3) and (4).

(d) For the spread:

Given a positive definite matrix A ,

$$sp(A) \leq \operatorname{tr} A - (n-1)\left(\frac{\det A}{\operatorname{tr} A}\right)^{1/n-1} - \left(\frac{n-1}{\operatorname{tr} A}\right)^{n-1} \det A \quad ;$$

and

$$\left(\frac{n}{\operatorname{tr} A}\right)^{n-1} \det A + (n-1)\left(\frac{n \det A}{\operatorname{tr} A}\right)^{1/n-1} - \operatorname{tr} A \leq sp(A) \quad .$$

□

(h) For the condition number:

For positive definite A ,

$$\frac{\left(\frac{n}{\operatorname{tr} A}\right)^{n-1} \det A}{\operatorname{tr} A - (n-1)\left(\frac{n \det A}{\operatorname{tr} A}\right)^{1/n-1}} \leq \frac{\lambda_1}{\lambda_n} = c(A)$$

$$\leq \frac{\operatorname{tr} A - (n-1)\left(\frac{\det A}{\operatorname{tr} A}\right)^{1/n-1}}{\left(\frac{n-1}{\operatorname{tr} A}\right)^{n-1} \det A} \quad .$$

□

§13:7.2

In this section we shall give results which involve the determinant and the diagonal elements of A. Since $0 < \max_i a_{ii} \leq \lambda_1$ and $0 < \lambda_n \leq \min_i a_{ii}$, inequalities (1) and (2) below, follow from 7.1 (1) and 7.1 (5).

(a) For $\lambda_1 = \max_i \lambda_i$:

Given positive semidefinite $A \neq 0$,

$$\left(\frac{n-1}{\operatorname{tr} A - \max_i a_{ii}}\right)^{n-1} \det A \leq \lambda_1 . \tag{1}$$

□

(b) For $\lambda_n = \min_i \lambda_i$:

Given positive definite A,

$$\lambda_n \leq \operatorname{tr} A - (n-1)\left(\frac{\det A}{\min_i a_{ii}}\right)^{1/n-1} . \tag{2}$$

□

The following result is given in [29]:

(h) For the condition number:

Given A positive definite, let

$$q = 4\, c(A) / (c(A) + 1)^2 .$$

Then:

$$q^{n-1} \prod_i a_{ii} \leq \det A , \tag{3}$$

with equality if and only if A is a scalar matrix. □

13:7.3.1

Given $\operatorname{tr} A$, $\operatorname{tr} A^2$ and $\det A$, for a positive definite matrix A , we have the following upper bound for the condition number:

(h) For the condition number:

Let A be positive definite, and $m = \operatorname{tr} A / n$ and $s^2 = \operatorname{tr} A^2 / n - m^2$, then:

$$\frac{\lambda_1}{\lambda_n} = c(A) \leq 1 + \frac{(2n)^{1/2} s [m + s/(n-1)^{1/2}]^{n-1}}{\det A} .$$

<u>Proof</u>: See [63]. □

PART II

SPECIALIZED TOPICS

CHAPTER 1

SPECTRAL RADIUS

§1:0 Preliminaries.

Given an $n \times n$ matrix A, let λ_i, $1 \leq i \leq n$ be its eigenvalues. Then,

$$\rho(A) = \max_i |\lambda_i| \tag{1}$$

is called the spectral radius of A. The bounds for $\rho(A)$ are given under the heading (a) in the chapters of Part I. In this chapter we shall study the relation between the spectral radius and the norm of A and give some results which are not included in Part I.

§1:1 Matrix Norms.

Given a square matrix A of order n, its matrix norm is a nonnegative number denoted by $||A||$, associated with A, such that:

(i) $||A|| \geq 0$, with equality if and only if $A = 0$;

(ii) $||cA|| = |c| \; ||A||$ for any scalar c;

(iii) $||A+D|| \leq ||A|| + ||D||$; and

(iv) $||AD|| \leq ||A|| \; ||D||$, where D is any matrix of order n.

However, matrices usually occur in conjunction with vectors and it is convenient to define a matrix norm so that it is compatible with a vector norm, in the following sense:

Definition 1: A matrix norm is said to be compatible with a vector norm $||x||$ if $||Ax|| \leq ||A|| \; ||x||$. □

The following result is well-known:

Theorem 2: Given a matrix A of order n, then

$$||A|| = \max_{||x||=1} ||Ax|| \,, \tag{1}$$

defines a matrix norm. □

Note that the norms on the right of (1) are vector norms.

Definition 3: A matrix norm defined by means of (1) is called, a matrix norm induced by or subordinate to the vector norm. □

Clearly any matrix norm which is induced by a vector norm is compatible. The most commonly used subordinate matrix norms are the ones associated with the 1,2 and ∞ vector norms. Below, we obtain their explicit representations:

Theorem 4. The matrix norms subordinate to the 1,2 and ∞ vector norms are

$$||A||_1 = \max_j \sum_i |a_{ij}| \quad ; \tag{2}$$

$$||A||_2 = \sigma_1 \quad , \tag{3}$$

where σ_1 is the largest singular value of A ;

$$||A||_\infty = \max_i \sum_j |a_{ij}| \quad . \tag{4}$$

Proof: Let $||x||_1 = 1$, that is $\sum_i |x_i| = 1$. Then,

$$\begin{aligned} ||Ax||_1 &= \sum_i |\sum_j a_{ij}x_j| \\ &\leq \sum_i \sum_j |a_{ij}|\ |x_j| = \sum_j \sum_i |a_{ij}|\ |x_j| \quad . \\ &\leq \max_j \sum_i |a_{ij}| \quad . \end{aligned} \tag{5}$$

Now, suppose that the maximum in (5) is attained for $j = k$. Let x be such that $x_i = 0$, $i \neq k$ and $x_k = 1$. Then,

$$||Ax||_1 = \sum_i |a_{ik}| \quad .$$

Hence from Theorem (2),

$$||A||_1 = \max_{||x||=1} ||Ax|| = \max_j \sum_i |a_{ij}| \quad ,$$

which establishes (2). To prove (3), we let $||x||_2 = (x,x) = 1$. Then

$$||Ax||_2^{\,2} = (Ax,Ax) = (x,A^*Ax) \quad ,$$

and from the Rayleigh's principle (see Theorem I:12:0(4), we conclude that $||Ax||_2^{\,2}$ is the largest eigenvalue of the positive semidefinite matrix A^*A . Finally since A^*A and AA^* have the same eigenvalues (see Theorem 4:0(1)), we conclude that

$$\max_{||x||=1} ||Ax||_2^{\,2} = \sigma_1^{\,2} \quad .$$

Now (3) follows from Theorem (2). Finally, to prove (4) let $||x||_\infty = \max_i |x_i| = 1$. Then,

$$||Ax||_\infty = \max_i |\sum_j a_{ij}\, x_j| \leq \max_i \sum_j |a_{ij}|\, |x_j|$$

$$\leq \max_i \sum_j |a_{ij}| \quad .$$

Thus,

$$||A||_\infty = \max_{||x||=1} ||Ax||_\infty \leq \max_i \sum_j |a_{ij}| \quad . \tag{6}$$

Now, let the maximum on the right in (6) be attained for $i = k$ and consider x such that $x_j = 1$ if $a_{kj} \geq 0$ and $x_j = -1$ if $a_{kj} < 0$. Then

$$||Ax||_\infty = \sum_j |a_{kj}| \quad ,$$

and (4) follows from Theorem (2). □

<u>Remark 5</u>. The $||A||_2$ is also called the <u>spectral norm</u>. □

<u>Definition 6</u>. A matrix norm $||\cdot||$, is called <u>unitarily invariant</u> if, from any unitary matrix U ,

$$||U^*AU|| = ||A|| \quad . \qquad \square$$

<u>Remark 7</u>. The Euclidean norm, $||A||^2 = \sum_{i,j} |a_{ij}|^2$, discussed in Chapter I:4 is not subordinate to any vector norm. For, in case of a subordinate matrix norm, by definition, the norm of the identity matrix is one, while the Euclidean norm is clearly $n^{1/2}$, where n is the order of the identity matrix. However, it is compatible with the Euclidean vector norm. Finally, the Euclidean and the spectral norms are unitarily invariant, while the matrix norms induced by 1 and ∞ vector norms are not. □

The following result is given in [42]:

<u>Theorem 8</u>. For any subordinate matrix norm $||\cdot||$,

$$\rho(A) \leq ||A|| \quad . \tag{7}$$

<u>Proof</u>: Let λ be any eigenvalue of A and x be the corresponding normalized eigenvector. Then,

$$||A|| \geq ||Ax|| = ||\lambda x|| = |\lambda| \; ||x|| = |\lambda| \quad ,$$

which completes the proof. □

By using the above theorem one can derive bounds for the spectral radius. For example from Theorem (4), we obtain $\rho(A) \leq \max_i R_i$, a result of I:6:0.

Given a normal matrix A, $\sigma_i = |\lambda_i|$, $1 \leq i \leq n$ (see Theorem 4:1 (2)). Thus, for a normal matrix A, equality in (7) holds for the spectral norm. The following theorems provide the necessary and sufficient conditions for equality in (7), in case of the spectral norm.

<u>Theorem 9</u>. Let A be an $n\times n$ matrix and $||\cdot||$ be the spectral norm. Then $\rho(A) = ||A||$ if and only if $||A^n|| = ||A||^n$.

<u>Proof</u>: See [49]. □

<u>Theorem 10</u>. Let A be an $n\times n$ matrix and $||\cdot||$ be the spectral norm. Then $\rho(A) = ||A||$ if and only if A is unitarily similar to a matrix of the form

$$\begin{pmatrix} \Lambda & 0 \\ 0 & B \end{pmatrix},$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_s \end{pmatrix}, \quad B = \begin{pmatrix} \lambda_{s+1} & & 0 \\ & \ddots & \\ (b_{ij}) & & \lambda_n \end{pmatrix},$$

$\rho(A)^2 I - B^* B$ is positive semidefinite, and s is such that $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_s| > |\lambda_{s+1}| \geq \cdots \geq |\lambda_n|$, and I is the identity matrix of order $n-s$.

<u>Proof</u>: See [14]. □

§1:2

In this section we shall give bounds for the spectral radius, which are not included in Part I.

The following theorem is well-known:

<u>Theorem 1</u>. Given an $n \times n$ matrix A,

$$\rho(A) \leq n \max_{i,j} |a_{ij}| \quad . \tag{1}$$

Further, if A is normal, then

$$\max_{i,j} |a_{ij}| \leq \rho(A) \quad . \tag{2}$$

<u>Proof</u>: For the spectral norm, Theorem 1 (8) gives, $\rho(A) \leq \sigma_1$. Thus, $\rho(A)^2 \leq \operatorname{tr} AA^* = \sum_{i,j} |a_{ij}|^2$ and we get $\rho(A)^2 \leq n^2 \max_{i,j} |a_{ij}|^2$, which proves (1). Since for normal A, $\rho(A) = \sigma_1$, the inequality (2) follows from 4:1 (2). □

<u>Theorem 2</u>. Let A be a $n \times n$ Hermitian matrix. Then

$$|\sum_{i,j} a_{ij}| / n \leq \rho(A) \quad . \tag{3}$$

When A is positive semidefinite,

$$\rho(A) \leq U_n \quad , \tag{4}$$

where,

$$U_1 = a_{11} \quad ,$$

$$U_j = \frac{1}{2}\left[a_{jj} + U_{j-1} + \sqrt{(a_{jj}-U_{j-1})^2 + 4\, b_j}\,\right] \quad ,$$

$$b_j = \sum_{i=1}^{j-1} |a_{ij}|^2 \quad , \quad j = 2,3,\cdots,n \quad \text{and}$$

λ_1 is the largest eigenvalue of A .

Proof: From the Rayleigh principle (see Theorem I:12:0 (3)),

$$\rho(A) \geq \max_{x \neq 0} (x,Ax)/(x,x) \quad .$$

Thus, choosing $x = (1,1,\cdots,1)'$, inequality (3) follows. It is also proved in [23]. Inequality (4) is proved in [52]. □

The following result is given in [5]:

Theorem 3: Let A be a positive definite matrix. Then

$$\operatorname{tr}(A^{k+1})/\operatorname{tr}(A^k) \leq \rho(A) \leq (\operatorname{tr}(A^k))^{1/k} \quad , \quad k = 1,2,\cdots,n,\cdots \quad . \tag{5}$$

Proof: Trivial. □

Next, we give some results for nonnegative matrices. We note that for any matrix A if $|A| = (|a_{ij}|)$ then $\rho(A) \leq \rho(|A|)$ (see Theorem I:10:0 (12)). Thus the results for a nonnegative matrix are also applicable to an arbitrary complex matrix.

We shall need the following notation:

$$R_i = \sum_j a_{ij} \quad , \quad P_i = R_i - a_{ii} \quad ,$$
$$R = \max_i R_i \quad \text{and} \quad r = \min_i R_i \quad . \tag{6}$$

The following theorem is proved in [8] and [9].

Theorem 4. Let A be an $n \times n$ nonnegative irreducible matrix. Then for $n > 2$,

$$\min_{i \neq j} M(i,j) \leq \rho(A) \leq \max_{i \neq j} M(i,j) \quad , \tag{7}$$

where

$$M(i,j) = \frac{1}{2} \{a_{ii} + a_{jj} + [(a_{ii} - a_{jj})^2 + 4\ P_i P_j]^{1/2}\} \quad .$$

Also,

$$\max_{i \neq j} \frac{1}{2} \{a_{ii} + a_{jj} + [(a_{ii} - a_{jj})^2 + 4\ a_{ij} a_{ji}]^{1/2}\} \leq \rho(A) \quad . \tag{8}$$

□

The inequalities (9) and (10) below are proved in [19] and [44], respectively.

Theorem 5. Let A be an $n \times n$ irreducible matrix. For fixed j let

$$\lambda_{ij} = \frac{1}{2} \{R_i - a_{ij} + a_{jj} + [(R_i - a_{ij} - a_{jj})^2 + 4\ a_{ij}(R_j - a_{jj})]^{1/2}\} \ ,$$

$$i = 1,2,\cdots,n \ , \ i \neq j \quad .$$

Then:

$$\min_{i \neq j} \lambda_{ij} \leq \rho(A) \leq \max_{i \neq j} \lambda_{ij} \quad . \tag{9}$$

Further, if $\alpha = \min a_{ii}$, $\sigma = \sum_i R_i/n$ and k is the least of the positive off-diagonal elements of A , then:

$$\frac{(n-1)(1-\epsilon)r + n\epsilon\sigma}{(n-1)(1-\epsilon) + n\epsilon} \leq \rho(A) \leq \frac{(n-1)(1-\epsilon)R + n\epsilon\sigma}{(n-1)(1-\epsilon) + n\epsilon} \quad , \tag{10}$$

where $\epsilon = (k / (R-\alpha))^{n-1}$, so that

$$r + \epsilon(\sigma - r) \leq \rho(A) \leq R - \epsilon(R-\sigma) \quad . \qquad \square$$

It is shown in [36] that if a nonnegative irreducible matrix is also symmetric then the lower bound for $\rho(A)$, given by (3) is better than those given by (7), (9) and (10). □

CHAPTER 2

SPREAD

§2:0 Preliminaries.

Given an nxn matrix A , with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, the maximum distance between the eigenvalues of A is called the spread of A , that is,

$$sp(A) = \max_{i,j} |\lambda_i - \lambda_j| \quad . \tag{1}$$

Also, we define

$$sp_R(A) = \max_{i,j} (Re(\lambda_i) - Re(\lambda_j)) \quad , \tag{2}$$

and

$$sp_I(A) = \max_{i,j} (Im(\lambda_i) - Im(\lambda_j)) \quad . \tag{3}$$

Bounds for the spread are given under the heading (d) in Part I. Here, we shall give three theorems regarding $sp(A)$, $sp_R(A)$ and $sp_I(A)$ and some results which are not included in Part I. Unless otherwise stated, in case of complex eigenvalues we shall assume that they are ordered as:

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \quad ,$$

and if all the eigenvalues are real then they are ordered as

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad .$$

Thus, if A is Hermitian,

$$sp(A) = sp_R(A) = \lambda_1 - \lambda_n \quad , \tag{4}$$

and

$$sp_I(A) = 0 \; .$$

The following two theorems are given in [40] and [41]:

Theorem 1. Let A be a normal $n\times n$ matrix. Then

$$sp(A) = \sup_{u,v} |u^* Au - v^* Av| \quad , \tag{5}$$

$$sp(A) \geq \sqrt{3} \sup_{u,v} |u^* Av| \quad , \tag{6}$$

where u and v are orthonormal vectors. Also,

$$sp(A) \geq \sup_{|z|=1} sp\left(\frac{zA + \bar{z}A^*}{2}\right) \quad . \tag{7}$$

□

When A is Hermitian, we have the following:

Theorem 2. Let A be a $n\times n$ Hermitian matrix. Then:

$$sp(A) = 2 \sup_{u,v} |u^* Av| \quad , \tag{8}$$

where u and v are othonormal vectors. □

The following theorem provides an upper bound for $sp_R(A)$ and $sp_I(A)$:

Theorem 3: Let A be an $n\times n$ matrix. Then:

$$sp(A) \leq sp(B) \quad , \tag{9}$$

$$sp_I(A) \leq sp(C) \quad , \tag{10}$$

where, $B = \frac{1}{2}(A+A^*)$ and $C = \frac{1}{2i}(A-A^*)$. Equality holds in (9) and (10) if A is normal.

<u>Proof</u>: Let the eigenvalues of B and C be $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$, respectively. Then from Theorems 4:2 (1) and 4:2 (2) we have,

$$\max_i \mathrm{Re}(\lambda_i) \leq \mu_1 \quad , \quad \mu_n \leq \min_i \mathrm{Re}(\lambda_i) \quad , \tag{11}$$

$$\max_i \mathrm{Im}(\lambda_i) \leq \nu_1 \quad \text{and} \quad \nu_n \leq \min_i \mathrm{Im}(\lambda_i) \quad . \tag{12}$$

Inequalities (9) and (10) follow from (11) and (12). Also from Theorem 11:0 (1), equality holds in (11) and (12) for normal A. □

§2:1

Here we shall give bounds for $sp(A)$, $sp_R(A)$ and $sp_I(A)$ of an $n\times n$ matrix A .

Theorem 1. Let A be a $n\times n$ Hermitian matrix. Then:

$$2 \max_{i\neq j} |a_{ij}| \leq sp(A) \quad ; \tag{1}$$

$$\max_{i\neq j} \{(a_{ii}-a_{jj})^2 + 4\, |a_{ij}|^2\}^{1/2} \leq sp(A) \quad ; \tag{2}$$

and

$$\frac{1}{2} \max_{i\neq j} \{a_{ii} + a_{jj} + [(a_{ii}-a_{jj})^2 + 4|a_{ij}|^2]^{1/2}\} - \\ - \frac{1}{2} \min \{a_{ii} + a_{jj} - [(a_{ii}-a_{jj})^2 + 4|a_{ij}|^2]^{1/2}\} \leq sp(A) . \tag{3}$$

Proof: Inequalities (1) and (2) are proved in [41]. Setting $u = e_i$ and $v = e_j$, $i \neq j$ in Theorem 0 (2), (1) follows. Further, if P is any principal submatrix of A , then from the Theorem I:12:0 (5),

$$sp(P) \leq sp(A) .$$

Thus, choosing,

$$P = \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix} ,$$

inequality (2) follows. Inequality (3) in proved in [10]. □

Theorem 2. Let A be a $n\times n$ Hermitian matrix. Define,

$$U_1 = L_1 = a_{11} \quad ,$$

$$U_j = \frac{1}{2}\left[a_{jj} + U_{j-1} + \sqrt{(a_{jj}-U_{j-1})^2 + 4\,b_j}\,\right] \quad ,$$

$$L_j = \frac{1}{2}\left[a_{jj} + L_{j-1} - \sqrt{(a_{jj}-L_{j-1})^2 + 4\,b_j}\,\right] \quad ,$$

$$b_j = \sum_{i=1}^{j-1} |a_{ij}|^2 \quad , \quad j = 2,3,\cdots,n \quad .$$

Then:

$$sp(A) \leq U_n - L_n \quad . \tag{4}$$

Proof: It is proved in [52] that $\lambda_1 \leq U_n$ and $\lambda_n \geq L_n$. Thus (4) follows. □

The following result is derived in [23]. It provides an algorithm for calculating the lower bound for the spread of a Hermitian or normal matrix. It is particularly good for a nonnegative symmetric matrix (see [23]).

Let I and J be non-empty disjoint subsets of $\{1,2,\cdots,n\}$ and,

$$K = \{1,2,\cdots,n\} \setminus (I \cup J) \quad .$$

Let s and t denote the cardinality of I and J, respectively. Then we have the following:

<u>Theorem 3</u>. If I and J are partitions of $\{1,2,\cdots,n\}$ as above, then:

$$\left|\frac{1}{t}\sum_{i,j\in I} a_{ij} - \frac{1}{s}\sum_{i,j\in J} a_{ij}\right| \leq sp(A) \quad , \tag{5}$$

if A is normal. In case A is Hermitian,

$$\frac{2}{\sqrt{st}}\left|\sum_{\substack{i\in I\\ j\in J}} a_{ij}\right| \leq sp(A) \quad . \tag{6}$$

□

<u>Theorem 4</u>. Let A be a normal matrix of order $n \geq 3$. Let s_1 be the trace of any principal minor matrix P of order $k \geq 3$ of A and s_2 the sum of principal minors of order 2 of P. Then:

$$sp(A) \geq \begin{cases} (\frac{2}{k})\ |(k-1)s_1{}^2 - 2k\ s_2|^{1/2} & , \quad k \text{ even} , \\ \\ \{\frac{4}{(k^2-1)^{1/2}}\}^{1/2}\ |(k-1)s_1{}^2 - 2k\ s_2|^{1/2} & , \quad k \text{ odd} . \end{cases} \tag{7}$$

<u>Proof</u>: See [10]. □

The following results are given in [40] and [41]:

<u>Theorem 5</u>. Let A be an $n\times n$ normal matrix. Then:

$$3^{1/2} \max_{i\neq j} |a_{ij}| \leq sp(A) \quad ; \tag{8}$$

$$\max_{i\neq j} \{(Re(a_{ii}) - Re(a_{jj}))^2 + |a_{ij} + \overline{a_{ji}}|^2\}^{1/2} \leq sp(A) \quad ; \tag{9}$$

$$\max_{i\neq j} \{|a_{ii}-a_{jj}|^2 + (|a_{ij}| - |a_{ji})^2\}^{1/2} \leq sp(A) \quad ; \tag{10}$$

$$\max_{i \neq j} (|a_{ij}| + |a_{ji}|) \leq sp(A) \quad ; \tag{11}$$

and

$$\max_{i \neq j} \left(\tfrac{1}{2} c_{ij}\right)^{1/2} \leq sp(A) \quad , \tag{12}$$

where,

$$c_{ij} = |a_{ii}-a_{jj}|^2 + |(a_{ii}-a_{jj})^2 + 4\, a_{ij}a_{ji}| + 2|a_{ij}|^2 + 2|a_{ji}|^2 \quad .$$

Proof: Choosing $u = e_i$ and $v = e_j$, $i \neq j$ in 0 (6), (8) follows. From 0 (7) and (2), with $|z| = 1$, we have for $i \neq j$,

$$(sp(A))^2 \leq \{Re(a_{ii}z) - Re(a_{jj}z)^2\}^2 + |a_{ij}z + \overline{a_{ji}z}|^2 \quad . \tag{13}$$

Choosing $z = 1$, we obtain (9). Also (13) implies

$$(sp(A))^2 \geq [Re\{(a_{ii}-a_{jj})z\}]^2 + (|a_{ij}|-|a_{ji}|)2 \quad . \tag{14}$$

Also, since

$$\sup_{|z|=1} [Re\{(a_{ii}-a_{jj})z\}]^2 = |a_{ii} - a_{jj}|^2 \quad ,$$

inequality (14) yields (10). Again, from (13),

$$sp(A) \geq \sup_{|z|=1} |a_{ij}z + \overline{a_{ji}z}| = |a_{ij}| + |a_{ji}| \quad ,$$

which proves (11). For the proof of (12) see [41]. □

Given A , upper bounds for $sp_R(A)$ and $sp_I(A)$ can be obtained using Theorem 0 (3) and any upper bound for the spread of a Hermitain matrix. In particular, Theorem (2) provides upper bounds for $sp_R(A)$ when a_{ij} is replaced by $b_{ij} = \frac{1}{2}(a_{ij} + \overline{a_{ji}})$ and for $sp_I(A)$ when a_{ij} is replaced by $c_{ij} = \frac{1}{2i}(a_{ij} - \overline{a_{ji}})$. Similarly, for normal A , Theorem 0 (3) can also provide lower bounds for $sp_R(A)$ and $sp_I(A)$.

CHAPTER 3

GERSCHGORIN DISKS

§3:0 Preliminaries.

Given an $n\times n$ matrix $A = (a_{ij})$ and $X(x) = \text{diag}(x_1,x_2,\cdots,x_n)$ where $x_i > 0$, let

$$\Lambda_i(x) = \Big(\sum_{j\neq i} |a_{ij}|\ x_j\Big) / x_i \quad , \quad i = 1,2,\cdots,n \quad . \tag{1}$$

Then the Gerschgorin disk $G_i(x)$ in the complex plane is defined by,

$$G_i(x) = \{z : |z-a_{ii}| \le \Lambda_i(x)\} \quad , \quad i = 1,2,\cdots,n \quad . \tag{2}$$

Considering the matrix $X^{-1}AX$, it follows from the Gerschgorin's Theorem (see I:6:2) that each eigenvalue of A lies in at least one of the disks $G_i(x)$, $1 \le i \le n$. Thus, the Gerschgorin set,

$$G(X) = \bigcup_i G_i(x) \quad , \tag{3}$$

contains all the eigenvalues of A. Further, if A is irreducible, an eigenvalue of A is boundary point of the union of the Gerschgorin disks only if it is the boundary point of all the disks. For, if λ is an eigenvalue of A such that $|\lambda-a_{ii}| \ge \Lambda_i(x)$, $i = 1,2,\cdots,n$, then it follows from I:6:2 ~~(10)~~, that equality must hold in all of the above inequalities, as $\det(\lambda I-A) = 0$.

Also, from Gerschgorin's Theorem (see I:6:2) if $\bigcup_{i=1}^{k} G_i$ has no point in common with the remaining $(n-k)$ disks, then $\bigcup_{i=1}^{k} G_i$ contains

exactly k eigenvalues of A . If fact, using continuity arguments one can prove the following (see e.g. [24]):

Theorem 1. Let J_s be any subset containing s elements of $J = \{1,2,\cdots,n\}$ and define $S(x) = \bigcup_{J_s} G_i(x)$ and $T(x) = \bigcup_{J/J_s} G_i(x)$. If

$$S(x) \cap T(x) \subset \partial G(x) \quad , \tag{4}$$

where $\partial G(x)$ is the boundary of $G(x)$, then $S(x)$ contains exactly s eigenvalues of A . □

Now let

$$d_{kj} = |a_{kk} - a_{jj}| \quad , \quad 1 \leq k \ , \ j \leq n \quad .$$

Then we have the following:

Definition 2. Let P_k be the set of all positive vectors x such that

$$d_{kj} = |a_{kk} - a_{jj}| \geq \Lambda_j(x) + \Lambda_k(x) \quad \text{for all} \quad j \neq k \quad . \tag{5}$$

If P_k is non-empty then we say that the matrix A admits, under diagonal similarity transformations, an isolated kth Gerschgorin disk. In case there exists a set J_s , $J_s \subset \{1,2,\cdots,n\}$ such that J_s has s elements and there exists $x > 0$ such that (5) holds for all $k \in S$ and $j \in \{1,2,\cdots,n\} / J_s$, we say that the s disks, J_s are isolated.

□

Now we show that if any s given disks are isolated as defined above then they contain exactly s eigenvalues of A. Without loss of generality we let $\bigcup_{i=1}^{k} G_i(x)$ be isolated. Then from the definition, for any $\ell \leq k$ and $k < m \leq n$, there exists a vector $x > 0$ such that

$$d_{\ell m} = |a_{\ell\ell}-a_{mm}| \geq \Lambda_\ell(x) + \Lambda_m(x) \quad . \tag{6}$$

Further, let $S(x) = \bigcup_{i=1}^{k} G_i(x)$ and $T(x) = \bigcup_{k+1}^{n} G_i(x)$.. If $S(x) \cap G(x)$ is empty then, from Theorem (1) the result follows. Let $z \in S(x) \cap T(x)$. We need to show that z is a boundary point.

Now, $z \in S(x) \cap T(x)$, implies there exists ℓ, $1 \leq \ell \leq k$ and m, $k < m \leq n$ such that

$$|z-a_{\ell\ell}| \leq \Lambda_\ell \quad \text{and} \quad |z-a_{mm}| \leq \Lambda_m \quad . \tag{7}$$

If possible, let equality not hold in one of the inequalities in (7). Then

$$\begin{aligned} d_{\ell m} = |a_{\ell\ell}-a_{mm}| &\leq |z-a_{\ell\ell}| + |z-a_{mm}| \\ &< \Lambda_\ell + \Lambda_m \\ &\leq d_{\ell m} \quad , \end{aligned}$$

which is a contradiction. Therefore $|z-a_{\ell\ell}| = \Lambda_\ell$ and $|z-a_{mm}| = \Lambda_m$. Hence z is a boundary point and from Theorem (1) $\bigcup_{i=1}^{k} G_i$ contains exactly k eigenvalues.

Naturally, in order to obtain bounds using Gerschgorin disks, if possible one would like to have isolated disks. This is one of the

reasons that the matrix $X^{-1}AX$ is considered instead of A to obtain the disks - it might happen that no disk of the matrix A $(X = I_n)$ is isolated but some disks of $X^{-1}AX$ are, for some choice of X. Secondly in case P_k is non-empty, that is, the kth disk is isolated, one would like to find the disk, with smallest possible radius μ,

$$\mu = \inf_{x \in P_k} \Lambda_k(x) \quad .$$

In the next section we briefly address the above mentioned, two questions and in section 2 we consider the case when A is real.

§3:1

In this section first we shall give results which provide sufficient conditions under which $G_k(x)$ can or cannot be isolated by some matrix $X = \mathrm{diag}(x_1, x_2, \ldots, x_n)$, $x_i > 0$, $1 \leq i \leq n$. These conditions are also necessary for certain class of matrices. As we shall see, in case the conditions for isolation are satisfied then $X(x)$ is readily determined. All these results are proved in [48]. We shall omit their proofs.

We shall consider the problem of isolating the kth disk only.

<u>Theorem 1</u>. Given an $n\times n$ matrix A, let

$$M_i = \max_{j\neq i} |a_{ij}| \quad , \quad t_k = \min_{j\neq k} d_{kj} = \min_{j\neq k} |a_{kk}-a_{jj}| > 0 \quad , \tag{1}$$

and

$$f_k(t) = \frac{M_k}{M_k+t} + \sum_{j\neq k} \frac{M_j}{M_j + d_{kj} - t} \quad . \tag{2}$$

Then, $G_k(x)$ can be isolated by some $X(x)$, and strict inequality holds in 0 (5), if $f_k(s) < 1$ for some s, $0 < s < t_k$.

<u>Proof</u>: See [48]. □

<u>Theorem 2</u>. Let the hypotheses of the above theorem be satisfied. If $M_j = 0$, for all j, $1 \leq j \leq n$ then $X(x) = I_n$ isolates G_k, with strict inequality in 0 (5). If $M_j \neq 0$ for some j's let M_{j_q}, $q = 1,2,\cdots,\ell$ be non-zero ($\ell \leq n$), and define

$$x_{j_1} = 1 \quad ,$$

$$x_{j_q} = \frac{\delta_{j_1}(s)}{\delta_{j_q}(s)} \quad , \quad q = 2,\cdots,\ell \quad ,$$

$$x_{j_q} = \frac{\alpha}{n+1-\ell} \quad , \quad q = \ell+1,\cdots,n \quad ,$$

where, $\alpha = \delta_{j_1}(s) - \sum_{q=1}^{\ell} x_{j_q}$, and

$$\delta_k(s) = \frac{M_k + s}{M_k} \quad ,$$

$$\delta_j(s) = \frac{M_j + d_{kj} - s}{M_j} \quad , \quad j \neq k \quad .$$

Then $X(x) = \mathrm{diag}(x_1, x_2, \cdots, x_n)$ isolates G_k , with strict inequality in 0 (5).

Proof: See [48]. □

Theorem 3. Let $m_i = \min_{j \neq i} |a_{ij}| > 0$ and t_k be given by (1). Define

$$g_k(t) = \frac{m_k}{m_k + t} + \sum_{j \neq k} \frac{m_j}{m_j + d_{kj} - t} \quad .$$

If $g_k(t) \geq 1$ for all t , $0 \leq t \leq t_k$, then no X isolates $G_k(x)$, with strict inequality in 0 (5).

Proof: See [48]. □

Theorem 4. Let A be such that $|a_{kj}| = \alpha_k$ for all $j \neq k$, $k = 1,2,\cdots,n$. Further, let t_k and f_k be given by (1) and (2), respectively. Then $G_k(x)$ can be isolated with strict inequality in 0 (5) by some $X(x)$ if and only if $f_k(s) < 1$ for some s , $0 < s < t_k$.

Proof: See [48]. □

Next, we give a convergent algorithm which, given an isolated kth disk, finds the smallest disk $G_k(x)$, with radius μ,

$$\mu = \inf_{x \in P_k} G_k(x) \quad ,$$

where P_k is as given by definition 0 (2). This result is proved in [60]. Its proof involves the theory of M-matrices. We shall state it without proof.

Without loss of generality we can assume that A is irreducible. Further if the kth Gerschgorin disk is isolated, that is, P_k is non-empty, we can assume that $k = 1$ (this can always be done by the use of a suitable permutation matrix) so that G_1 is isolated. Finally, as $\Lambda_j(hx) = h\, \Lambda_j(x)$ for $h > 0$ and $1 \le j \le n$, we let $x_1 = 1$, for all $x \in P_1$.

If $Q = (q_{ij})$ is a real matrix, such that

$$q_{ii} = d_{1i} = |a_{11} - a_{ii}| \quad , \quad i = 1, 2, \cdots, n \quad ,$$

$$q_{1j} = |a_{1j}| \quad , \quad 2 \le j \le n \quad ,$$

$$q_{ij} = -|a_{ij}| \quad , \quad i \ne j \; , \; i \ne 1 \quad .$$

Then clearly A irreducible implies that Q is irreducible. Further, we partition Q as,

$$Q = \begin{pmatrix} 0 & \hat{\alpha}' \\ -\hat{a} & \tilde{Q} \end{pmatrix} ,$$

where $\tilde{Q}$ is the principal submatrix of Q of order $(n-1)$, $\hat{\alpha}' = (|a_{12}|,|a_{13}|, \cdots ,|a_{1n}|)$ and $\hat{a}' = (|a_{21}|,|a_{31}|, \cdots ,|a_{n1}|)$. Let $\hat{y}$ be the vector with $(n-1)$ components obtained from the column vector $y = (y_i)$, by deleting y_1. Conversely given $\hat{y}$ let $y = (y_i)$ denote the unique column vector such that $y_1 = 1$, $y_{i+1} = \hat{y}_i$, $i = 1,2,\cdots,n-1$. Then we have the following:

<u>Theorem 5</u>. Let A be an irreducible matrix of order n, which admits a first isolated disk, $G_1(x)$. Then the smallest radius μ,

$$\mu = \inf_{x \in P_1} \Lambda_1(x) ,$$

for this isolated disk is an eigenvalue of the matrix Q and its corresponding eigenvector y_μ is uniquely determined in P_1. If $x_0 \in P_1$ and $Q\, x_0 \geq \Lambda_1(x_0)x_0$, with strict inequality in at least one component, then the sequence of vectors $\{x_i\}$ defined by

$$(\tilde{Q}\ \Lambda_1(x_i)I_{n-1})\ \hat{x}_{i+1} = \hat{a} \quad , \quad i \geq 0 \ ,$$

are all elements of P_1 with $\lim_{i\to\infty} x_i = y_\mu$, and the sequence $\{\Lambda_i(x_i)\}_{i=0}^{\infty}$ is strictly decreasing with $\lim_{i\to\infty} \Lambda_1(x_i) = \mu$.

<u>Proof</u>: See [60]. □

It is shown in [34], that the above algorithm can be used to directly estimate the isolated eigenvalue of A, rather than just obtaining the Gerschgorin disk of smallest possible radii. Also in [35] the above algorithm is extended to the case when more than one isolated disks

are given. Finally, in [24] an algorithm is given which does not depend on a prior knowledge that a given set of disks is isolated.

§3:2

In this section we shall give some results for a real matrix.

The following result is well-known (see e.g. [57, pg. 287]):

Theorem 1. Let A be a real matrix. If the Gerschgorin disk $G_k(x)$, $1 \leq k \leq n$, has no point in common with the remaining $(n-1)$ disks (or strict inequality holds in 0 (5)) then A has a real eigenvalue of multiplicity one.

Proof: From Gerschgorin's Theorem (see I:6:2), G_k has exactly one eigenvalue, say λ. In case λ is complex, since A is real, we conclude that $\bar{\lambda}$ is also an eigenvalue of A. Now λ and $\bar{\lambda}$ lie symmetrically about the real axis (since a_{kk} is real) and we have that both λ and $\bar{\lambda}$ belong to $G_k(x)$, which is a contradiction. Thus λ must be real.

The following result is immediate from the above theorem:

Theorem 2. Let A be a real matrix. If A has k Gerschgorin disks which have no point in common with any other disk, then A has k real distinct eigenvalues. □

The following theorem is proved in [48]:

Theorem 3. Let A be a real matrix and J_k a subset of $\{1,2,\cdots,n\}$ with n elements. If the ith Gerschgorin disk of $X_i^{-1}AX_i$ is isolated with strict inequality in 0 (5), for all $i \in J_k$, then A has at least k real distinct eigenvalues. □

CHAPTER 4

SINGULAR VALUES

§4:0 Preliminaries.

In this chapter we shall discuss the relationships among the eigenvalues of AA^* (square of singular values), $\frac{1}{2}(A+A^*)$ (real singular values), $\frac{1}{2i}(A-A^*)$ (imaginary singular values) and A, where A is an $n\times n$ complex matrix. We shall also include bounds for the singular bounds for the singular values, which are not included in Part I. Unless otherwise stated we shall assume that the eigenvalues of A are ordered as

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \quad ,$$

if complex and as

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad ,$$

if real.

As $(AA^*)^* = AA^*$ we conclude that AA^* is Hermitian. In fact, AA^* is positive semidefinite, since for any vector x $(AA^*x,x) = (A^*x,A^*x) \geq 0$. Thus, the eigenvalues of AA^* are nonnegative. Furthermore, it follows, from Theorem (1) below, that AA^* is positive definite if and only if A is nonsingular. Similarly, A^*A is positive semidefinite Actually, AA^* and A^*A have the same eigenvalues:

<u>Theorem 1</u>. Matrices AA^* and A^*A have the same eigenvalues.

<u>Proof</u>: Since A is $n\times n$ the proof follows immediately from the fact that

if H and K are matrices of order $m \times n$ and $n \times m$ respectively $(m \leq n)$ then the eigenvalues of KH are the m eigenvalues of HK and zero, repeated $n-m$ times (see [39, pg. 200]). □

<u>Definition 2</u>. If $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_n^2$ are the nonnegative ordered eigenvalues of AA^* then $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ are called the <u>singular values</u> of A. □

<u>Remark 3</u>. If A is rectangular of order $m \times n$ $(m \leq n)$ then A has m singular values. However, we shall generally consider only square matrices. Analogous results will hold for rectangular matrices. □

Now we state the Singular value decomposition theorem. The proof can be found in [42, pg. 330].

<u>Theorem 4</u>. If A is an $m \times n$ $(m \leq n)$ matrix, then there exist unitary matrices U, $m \times m$ and V, $n \times n$ such that

$$A = U\,D\,V \quad , \tag{1}$$

where $D = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_m)$ is $m \times n$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$ are the singular values of A. Further, if the rank of A is k then exactly k singular values are positive. □

Next define,

$$B = \frac{1}{2}(A+A^*) \quad \text{and} \quad C = \frac{1}{2i}(A-A^*) \quad . \tag{2}$$

Clearly B and C are Hermitian. Given A, it can be decomposed as

$A = B + iC$. It can be shown that this Hermitian decomposition is unique.

Definition 5. The eigenvalues of $B = (b_{ij})$ and $C = (c_{ij})$ are called the real and imagainary singular values of A , respectively. □

Remark 6. B and C will always be assumed to be defined by (2). Also, it will be assumed that

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \quad \text{and} \quad \nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \tag{3}$$

are the ordered eigenvalues of B and C . □

Remark 7. If A is Hermitian, then $B = A$ and $C = 0$, while if A is skew-Hermitian then $B = 0$ and $C = A$. □

Remark 8. It follows from the definition of B and C that,

$$\text{tr } B^2 = \frac{1}{2}(\text{tr } AA^* + \text{Re}(\text{tr } A^2)) \text{ and tr } C^2 = \frac{1}{2}(\text{tr } AA^* - \text{Re}(\text{tr } A^2)) . \tag{4}$$

□

§4:1 Singular Values and Eigenvalues.

We shall now study the relationship between the singular values and the eigenvalues of a matrix A.

Theorem 1. For any square $n \times n$ matrix A:

$$\prod_{1}^{k} |\lambda_i| \le \prod_{1}^{k} \sigma_i \quad , \quad k = 1,2,\cdots,n \quad , \tag{1}$$

with equality for $k = n$;

$$\sigma_n \le |\lambda_n| \quad ; \tag{2}$$

$$\sum_{1}^{k} |\lambda_i|^s \le \sum_{1}^{k} \sigma_i^s \quad , \quad s > 0 \ , \ k = 1,2,\cdots,n \ . \tag{3}$$

Equality holds in (1) for $k = 1,2,\cdots,n$ if and only if equality holds in (3) for $k = 1,2,\cdots,n$ if and only if A is normal.

Proof: To establish (2), we have, using the Rayleigh quotient, that $(AA^*x,x) \ge \sigma_n^2$, for any x such that $x^*x = 1$. Thus, if x is a normalized eigenvector of λ_n, we obtain

$$\sigma_n^2 \le (AA^*x,x) = (A^*x,A^*x) = (\bar{\lambda}_n x,\bar{\lambda}_n x)$$

$$= |\lambda_n|^2 (x,x) = |\lambda_n|^2 \quad ,$$

hence (2) follows. For the proof of (1) and (3) see [31, pg. 115]. Equality conditions follow from the theorem below. □

The following result is well-known (e.g. see [16]).

Theorem 2. Given A, it is normal if and only if,

$$\sigma_i = |\lambda_i| \quad , \quad i = 1,2,\cdots,n \quad . \tag{4}$$

Proof: Let A be normal. Then there exists a unitary matrix U such that $A = UDU^*$, $D = \text{diag}(\lambda_1,\lambda_2,\cdots,\lambda_n)$. Therefore $AA^* = UD\bar{D}U^*$ and, using the fact that similar metrices have the same eigenvalues, (4) follows. Conversely, let (4) hold. Then from Schur's triangularization theorem, there exists a unitary matrix U and an upper triangular matrix T such that $AA^* = UTT^*U^*$ and $t_{ii} = \lambda_i$, $i = 1,2,\cdots,n$. Thus, we conclude that AA^* and TT^* have the same eigenvalues and,

$$\sum_i \sigma_i^2 = \sum_i |\lambda_i|^2 + \sum_{i \neq k} |t_{ik}|^2 \quad .$$

From which we conclude that T is diagonal. Hence A is normal. □

The following result is immediate from the above theorem:

Corollary 3. If A is positive semidefinite then:

$$\sigma_i = \lambda_i \quad , \quad i = 1,2,\cdots,n \quad . \tag{5}$$

□

§4:2 The Matrices A, B and C.

In this section we shall establish relationships between the eigenvalues (λ_i's) of A and the eigenvalues (μ_i's and ν_i's) of B and C respectively. In particular, we shall see that the real (imaginary) singular values of A, majorize the real (imaginary) parts, $\mathrm{Re}(\lambda_i)$ $(\mathrm{Im}(\lambda_i))$ of A.

Theorem 1. Let the eigenvalues and real singular values of A be λ_i and μ_i, $i=1,2,\cdots,n$, respectively. Let $\mathrm{Re}(\lambda_1) \geq \mathrm{Re}(\lambda_2) \geq \cdots \geq \mathrm{Re}(\lambda_n)$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Then:

$$\mathrm{Re}(\lambda_1) \leq \mu_1 \quad ; \tag{1}$$

$$\mu_n \leq \mathrm{Re}(\lambda_n) \quad ; \tag{2}$$

$$\sum_1^k \mathrm{Re}(\lambda_i) \leq \sum_1^k \mu_i \quad , \quad k=1,2,\cdots,n \quad , \tag{3}$$

with equality in (3) for $k = n$.

Proof: Assertions (1) and (2) follows from the Rayleigh quotient. For if x is an eigenvector corresponding to the eigenvalue λ of A, then $\mu_n \leq \mathrm{Re}(\lambda) = (Bx,x) \leq \mu_1$. For the proof of (3) see [32, pg. 237]. $\square$

As one would expect, similar relations hold between the imaginary parts of the eigenvalues of A and the imaginary singular values of A:

Theorem 2. Let the eigenvalues and imaginary singular values of A be arranged so that $\operatorname{Im}(\lambda_1) \geq \operatorname{Im}(\lambda_2) \geq \cdots \geq \operatorname{Im}(\lambda_n)$ and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$. Then we have the following:

$$\operatorname{Im}(\lambda_1) \leq \nu_1 \quad ; \tag{4}$$

$$\nu_n \leq \operatorname{Im}(\lambda_n) \quad ; \tag{5}$$

$$\sum_1^k \operatorname{Im}(\lambda_i) \leq \sum_1^k \nu_i \quad , \quad k = 1,2,\cdots,n \quad , \tag{6}$$

with equality for $k = n$. □

The following result relating the eigenvalues of A, B and C is given in [16]:

Theorem 3. Given a matrix A, then $\operatorname{Re}(\lambda_i)$, $i = 1,2,\cdots,n$ are the eigenvalues of B if and only if $\operatorname{Im}(\lambda_i)$, $i = 1,2,\cdots,n$ are the eigenvalues of C, if and only if A is normal. □

§4:3 The Matrices AA^*, B and C.

The following three theorems exhibit the relationships among the three types of singular values (see e.g. [1]).

Theorem 1. If σ_i, μ_i, ν_i, $i = 1,2,\cdots,n$ are the singular, real singular, and imaginary singular values of A, respectively, then:

$$\operatorname{tr} B^2 + \operatorname{tr} C^2 = \sum_i \mu_i^2 + \sum_i \nu_i^2 = \sum_i \sigma_i^2 = \operatorname{tr} AA^* \quad . \tag{1}$$

Proof: The result follows from expanding

$$\operatorname{tr} B^2 + \operatorname{tr} C^2 = \frac{1}{4} \operatorname{tr} (A+A^*)^2 - \frac{1}{4} \operatorname{tr} (A-A^*)^2 \quad ,$$

and using the fact that $\operatorname{tr} AA^* = \operatorname{tr} A^*A$ and that B and C are Hermitian. □

Theorem 2. For any $n \times n$ matrix A :

$$\mu_i \le \sigma_i \quad , \quad i = 1,2,\cdots,n \quad , \tag{2}$$

$$\nu_i \le \sigma_i \quad , \quad i = 1,2,\cdots,n \quad . \tag{3}$$

Proof: The inequality (2) is proved in [13]. If we replace A by $-iA$, then the σ_i, $i = 1,2,\cdots,n$, are unchanged but the real singular values of $-iA$ are ν_i, $i = 1,2,\cdots,n$. Thus (3) follows from (2). □

Theorem 3. If A is normal, then:

$$\sigma_1^2 \le \max_i \mu_i^2 + \max_i \nu_i^2 \quad ; \tag{4}$$

$$\sigma_n^2 \ge \min_i \mu_i^2 + \min_i \nu_i^2 \quad . \tag{5}$$

Proof: From Theorems 1 (2) and 2 (3) we have

$$\sigma_1^2 = |\lambda_1|^2 = \mu_j^2 + \nu_k^2 \quad \text{and} \quad \sigma_n^2 = \mu_\ell^2 + \nu_m^2 \quad ,$$

for some $1 \le j,k,\ell,m \le m$. Thus, taking the maximum and minimum of μ_i^2 and ν_i^2 , $i = 1,2,\cdots,n$, (4) and (5) follow. □

§4:4 Bounds for Singular Values.

Bounds for the singular values, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$, are given under the heading (g) in the chapters of Part I. Also, since σ_i^2 , $i = 1,2,\cdots,n$ are the eigenvalues of the positive semidefinite matrix AA^* , we can apply the corresponding results from Chapter 13 in Part I. Below, we give results which are not included in Part I (we attempt to maintain the same ordering of the information as used in Part I).

(a) For σ_1 :

Theorem 1. Given A ,

$$\max_{i,j} |a_{ij}|^2 \leq \max_i \sum_j |a_{ij}|^2 \leq \max_{i,j} |\sum_k a_{ik} \overline{a_{jk}}| \leq \sigma_1^2 \quad ; \tag{1}$$

$$\max_{s,t} \frac{1}{\sqrt{st}} \left| \sum_{j=1}^{t} \sum_{i=1}^{s} a_{ij} \right| \leq \sigma_1 \quad , \quad 1 \leq s \, , \; t \leq n-1 \; . \tag{2}$$

Proof: The first two inequalities in (1) are clear. To prove the inequality on the right-hand side of (1) we consider AA^* . Since AA^* is positive semidefinite, its eigenvalues σ_i^2 , $i = 1,2,\cdots,n$ are also its singular values (see Corollary 1 (3)). Thus, choosing $t_1 = 1$, $t_2 = t_3 = \cdots = t_n = 0$ in Theorem (7) below, we obtain $\sigma_1^2 \geq u^*AA^*v$, where $||u|| = ||v|| = 1$. Therefore with $u = e_i$ and $v = e_j$, (1) follows. This result is also proved in [47]. To prove (2), once again Theorem (7) implies $\sigma_1 \geq u^*Av$ with $||u|| = ||v|| = 1$. Choosing $u = (u_i)$ and $v = (v_i)$ such that $u_i = 1/\sqrt{s}$, $i = 1,2,\cdots,s$, $u_i = 0$ for $i > s$, $u_i = 1/\sqrt{t}$, $i = 1,2,\cdots,t$ and $v_i = 0$ for $i > t$, (2) follows. We note

that u and v could be choosen differently, to obtain bounds different from (1) and (2). □

The following result is given in [7, pg. 67]:

Theorem 2. For real A,

$$\sigma_1 \leq \max_i \sum_j |b_{ij}| + \max_i \sum_j |c_{ij}| \quad , \tag{3}$$

where $B = (b_{ij}) = \frac{1}{2}(A+A')$ and $C = (c_{ij}) = \frac{1}{2i}(A-A')$. □

(b) For σ_n :

An interesting lower bound for σ_n is given in [22]:

Theorem 3. Let A be an $m{\times}n$ $(n \leq m)$ matrix and $s = \{s_2, s_3, \cdots, s_n\}$ be a set of positive column scaling weights such that

$$\sum_{i=1}^{n} s_i^2 = 1 \ .$$

Then:

$$\min_j \sum_{i=1}^{m} w_{ij}^2 \leq \sigma_n^2 \quad , \tag{4}$$

where

$$w_{ij} = \max(0\ , |a_{ij}|\, s_j - \sum_{k\neq j} |a_{ik}| s_k) \ . \tag{5}$$

□

We can modify the above theorem in the case of a normal matrix A :

Theorem 4. If A is normal and $s_1, s_2, \cdots, s_n$ are positive numbers such that

$$\sum_i s_i^2 = 1 \ .$$

Then:

$$\min_j \sum_i p_{ij}^2 + \min_j \sum_i q_{ij}^2 \leq \sigma_n^2 \ , \tag{6}$$

where

$$p_{ij} = \max(0 \, , \frac{1}{2} \, |a_{ij} + \overline{a_{ji}}| \, s_j - \frac{1}{2} \sum_{k \neq j} |a_{ik} + \overline{a_{ki}}| \, s_k) \ , \tag{7}$$

and,

$$q_{ij} = \max(0 \, , \frac{1}{2} \, |a_{ij} - \overline{a_{ji}}| \, s_j - \frac{1}{2} \sum_{k \neq j} |a_{ik} - \overline{a_{ki}}| \, s_k) \ , \quad i,j = 1,2,\cdots,n. \tag{8}$$

Proof: From inequality 3 (5) we have $\sigma_n^2 \geq \min_i |\mu_i|^2 + \min_i |\nu_i|^2$ and we observe that $\min_i |\mu_i|$ and $\min_i |\nu_i|$ are the singular values of B and C respectively. Let p_{ij} and q_{ij}, $i = 1,2,\cdots,n$ be as given by (7) and (8) respectively. Then, applying Theorem (3) to B and C, we have,

$$\min \mu_i^2 \geq \min_j \sum_i p_{ij}^2 \quad \text{and} \quad \min \nu_i^2 \geq \min_j \sum_i q_{ij}^2 \ .$$

Thus, we get

$$\sigma_n^2 \geq \min_j \sum_i p_{ij}^2 + \min_j \sum_i q_{ij}^2 \ ,$$

which completes the proof. □

The following lower bound for σ_n is given in [7, pg. 67]:

Theorem 5. For real A,

$$\min(|a_{ii}| - \sum_k |b_{ik}|) - \max \sum_k |c_{ik}| \leq \sigma_n \quad , \tag{9}$$

where, $B = \frac{1}{2}(A+A')$ and $C = \frac{1}{2i}(A-A')$. □

The following upper bound for σ_n is well-known.

Theorem 6. Given A,

$$\sigma_n \leq \min_i (\sum_j |a_{ij}|^2)^{1/2} \leq \min_i \sum_j |a_{ij}| \quad . \tag{10}$$

Proof: From the Rayleigh quotient $\sigma_n^2 \leq x^* AA^* x$, $||x|| = 1$. Hence choosing $x = e_i$, (10) follows. □

(e) For sum of singular values:

The following theorem provides an algorithm for obtaining lower bounds of nonnegative linear combinations of singular values of A. The proof is similar to the one used in Theorem 6 of [41], for obtaining the bounds for the spread of a matrix.

Theorem 7. If $t_1 \geq t_2 \geq \cdots \geq t_n \geq 0$, then:

$$\sum_{i=1}^{k} t_i \sigma_i = \sup_{u_i, v_i} \sum_{i=1}^{k} t_i |u_i^* A v_i| \quad , \quad k = 1,2,\cdots,n \quad , \tag{11}$$

where u_i and v_i , $i = 1,2,\cdots,n$ are two sets of orthonormal vectors. In particular:

$$\sum_1^k |a_{ii}| \le \sum_1^k \sigma_i \quad , \quad k = 1,2,\cdots,n \quad ; \tag{12}$$

$$\sum_{i=1}^k |a_{i\ n-i+1}| \le \sum_1^k \sigma_i \quad , \quad k = 1,2,\cdots,n \quad . \tag{13}$$

Proof: Let the moduli of the diagonal elements of A be denoted by $d_1 \ge d_2 \ge \cdots \ge d_n$. Then from I:2:0 (7),

$$\sum_1^k di \le \sum_1^k \sigma_i \quad , \quad k = 1,2,\cdots,n$$

and we have

$$(t_k - t_{k+1}) \sum_1^k d_i \le (t_k - t_{k+1}) \sum_1^k \sigma_i \quad , \quad k = 1,2,\cdots,n-1 \quad ,$$

$$t_n \sum_i di \le t_n \sum_i \sigma_i \quad .$$

Adding the above set of inequalities yields

$$\sum_i t_i\ d_i \le \sum_i t_i\ \sigma_i \quad . \tag{14}$$

Now, let $U = [u_1,u_2,\cdots,u_n]$ and $V = [v_1,v_2,\cdots,v_n]$ be unitary matrices with columns u_i and v_i , respectively. Then the matrix $U A V$ has the same singular values as A . Thus, if we order u_i , v_i , $i = 1,2,\cdots,n$ such that $|u_1^* A v_1| \ge |u_2^* A v_2| \ge \cdots \ge |u_n^* A v_n|$, from (14), we obtain

$$\sum_i t_i\ \sigma_i \ge \sum_i t_i\ |u_i^* A v_i| \quad .$$

Hence,

$$\sum_i t_i \sigma_i \geq \sup_{\substack{u_i, v_i \\ i=1,2,\cdots,n}} \sum_i t_i |u_i^* A v_i| \quad . \tag{15}$$

Further, equality in (15) is attained for U and V such that $\operatorname{diag}(\sigma_1,\cdots,\sigma_n) = UAV$. Thus (11) follows. With $t_1 = t_2 = \cdots = t_k = 1$, $t_{k+1} = \cdots = t_n = 0$, inequality (12) follows by choosing $u_i = v_i = e_i$, $i = 1, 2, \cdots, k$, while (13) follows by choosing $u_i = e_i$ and $v_i = e_{n-i+1}$, $i = 1, 2, \cdots, k$. □

Below we present another lower bound for the partial sums of singular values.

Theorem 8. Let A_k denote any $k \times k$ submatrix of A obtained by deleting $(n-k)$ rows and $(n-k)$ columns of A , and let t_k be the sum of the absolute values of the elements of A_k . Then:

$$\max_k \frac{1}{k} t_k \leq \sum_1^k \sigma_i \quad , \quad k = 1,2,\cdots,n-1 \quad . \tag{16}$$

Proof: By the Singular value decomposition theorem, there exist $k \times k$ unitary matrices U and V such that $A_k = UDV$, where $D = \operatorname{diag}(\sigma_1',\cdots,\sigma_k')$ and σ_i' , $i = 1,2,\cdots,k$ are the singular values of A_k . Thus,

$$a_{pq} = \sum_{m=1}^k u_{pm} \sigma_m' v_{mq} \quad , \quad \text{i.e.} \quad t_k \leq \sum_{m=1}^k \sigma_m' \sum_{p=1}^k |u_{pm}| \sum_{q=1}^k |v_{mq}| \quad ,$$

which implies, $\frac{1}{k} t_k \leq \sum_1^k \sigma_m'$, since $\left(\sum_{p=1}^k |u_{pm}|\right)^2 \leq k \sum_{p=1}^k |u_{pm}|^2 = k$ and $\left(\sum_{q=1}^k |v_{mq}|\right)^2 \leq k$.

Lastly, as σ_i', $i = 1,2,\cdots,k$ are the singular values of A_k, we have $\sum_{i=1}^{k} \sigma_i' \leq \sum_{1}^{k} \sigma_i$ (see [55]). This completes the proof of (16).

Finally, we give a lower bound for the partial sums of the squares of the singular values:

<u>Theorem 9</u>. Let s_i, $i = 1,2,\cdots,n$ be the squared sums $\sum_j |a_{ij}|^2$, arranged in decreasing order. Then

$$\sum_{i=1}^{k} s_i \leq \sum_{i=1}^{k} \sigma_i^2 \quad , \quad k = 1,2,\cdots,n \quad . \tag{17}$$

Equality holds for $k = n$.

<u>Proof</u>: The inequality (17) follows at once by applying the fact that eigenvalues of a Hermitian matrix majorize the diagonal elements, to the matrix AA^*. $\square$

§4:5 Bounds for Real and Imaginary Singular Values.

Since B and C are Hermitian, the bounds given in Chapter I : 12 are applicable to B and C (with a_{ij} replaced by $b_{ij} = \frac{1}{2}(a_{ij} + \overline{a_{ji}})$ for B and by $c_{ij} = \frac{1}{2i}(a_{ij} - \overline{a_{ji}})$ for C). One may also use Theorem 2 (3) if A is normal and Remark 0 (7) if A is Hermitian. □

CHAPTER 5

CONDITION NUMBER

§5:0 Preliminaries.

Consider the linear system

$$Ax = b \quad , \tag{1}$$

where, A is an $n \times n$ nonsingular matrix and x and b are $n \times 1$ vectors. Then (1) has a unique solution,

$$x = A^{-1} b \quad . \tag{2}$$

We shall now see how the solution (2) is affected by pertubations in the right-hand side of (1) and in the elments of the matrix A . Suppose that in (1) the vector b is perturbed to $b+\delta b$, and that the exact solution of the perturbed system is $x+\delta x$, that is,

$$A(x+\delta x) = b + \delta b \quad .$$

Therefore,

$$x + \delta x = A^{-1}(b + \delta b) \quad .$$

Using (2), we obtain,

$$\delta x = A^{-1} \delta b \quad .$$

Thus, for any compatible matrix norm, $||\cdot||$,

$$||\delta x|| \leq ||A^{-1}|| \; ||\delta b|| \quad . \tag{3}$$

Also from (1),

$$||b|| \leq ||A||\ ||x|| \quad . \tag{4}$$

Combining (3) and (4), we have,

$$\frac{||\delta x||}{||x||} \leq ||A||\ ||A^{-1}||\ \frac{||\delta b||}{||b||} \quad . \tag{5}$$

Similarly, if A is perturbed by δA and δx is the corresponding change in the solution vector x, then

$$(A+\delta A)(x+\delta x) = b \quad ,$$

which gives,

$$\delta x = -A^{-1}\ \delta A(x+\delta x) \quad \text{and}$$

$$||\delta x|| \leq ||A^{-1}||\ ||\delta A||\ ||x+\delta x|| \ ,$$

or

$$\frac{||\ x||}{||x+\delta x||} \leq ||A^{-1}||\ ||\delta A|| \ .$$

Considering the change $||\delta A||$ relative to $||A||$, we get

$$\frac{||\delta x||}{||x+\delta x||} \leq ||A||\ ||A^{-1}||\ \frac{||\delta A||}{||A||} \quad . \tag{6}$$

Thus, both in (5) and (6), the relative change in the exact solution is bounded by the number $||A||\ ||A^{-1}||$ multiplied by the relative perturbation in the data. The number $||A||\ ||A^{-1}||$ is called the <u>condition</u> <u>number</u> of A. We shall denote it by $c(A)$.

The condition number of A indicates the maximum effect of perturbations in b or A on the exact solution of (1). From (5) and (6) we see that if $c(A)$ is large, then the change in the exact solution of (1) may be large, even for small perturbations in b or A. Given a nonsingular matrix A, it is called <u>ill-conditioned</u> if $c(A)$ is large and <u>well-conditioned</u> if $c(A)$ is small.

Note that if for any matrix norm $||\cdot||$, $||I|| \geq 1$, then

$$1 \leq ||I|| \leq ||A|| \; ||A^{-1}|| = c(A) \; .$$

In particular, for any subordinate matrix norm and the Frobenius norm,

$$c(A) \geq 1 \; . \tag{7}$$

Further, for the spectral norm,

$$c(A) = \frac{\sigma_1}{\sigma_n} \geq 1 \; . \tag{8}$$

It follows from the Singular value decomposition theorem (see Theorem 4:0 (4)) that equality holds in (8) if and only if $A = cU$, for some unitary U and scalar c. Finally, we give the following result:

<u>Theorem 1</u>. Let A be a nonsingular matrix. Then for any subordinate matrix norm, $||\cdot||$,

$$\frac{\max_i |\lambda_i|}{\min_i |\lambda_i|} = \frac{|\lambda_1|}{|\lambda_n|} \leq c(A) = ||A|| \; ||A^{-1}|| \; .$$

<u>Proof</u>: From Theorem 1:1 (8), we have that, $\max_i |\lambda_i| \leq ||A||$. Further since the eigenvalues of A^{-1} are λ_i^{-1} , $i = 1,2,\cdots,n$, we again have

$$\max_i |\lambda_i^{-1}| = \frac{1}{\min_i |\lambda_i|} \leq ||A^{-1}|| \quad ,$$

which completes the proof. □

§ 5:1

Bounds for the condition number are given in Part I under the heading (h), for the spectral norm. However, in view of Theorem 0 (1) a lower bound for the condition number which is obtained by using the inequality

$$\frac{|\lambda_1|}{|\lambda_n|} \leq c(A) = \frac{\sigma_1}{\sigma_n} \quad ,$$

is actually a lower bound for any condition number defined by using a subordinate norm. Below we give a lower bound for the condition number and a technique for estimating it.

The following result is given in [27, pg. 29].

<u>Theorem 1</u>. Let A be a nonsingular, triangular matrix. Then for the spectral norm,

$$\frac{\max\limits_{i,j} |a_{ij}|}{\min\limits_{i} |a_{ii}|} \leq c(A) \quad . \tag{1}$$

<u>Proof</u>: It follows from Theorem II:4:4 (1) that $\max\limits_{i,j} |a_{ij}| \leq \sigma_1$. Also, as A is triangular, the diagonal elements of A^{-1} are a_{ii}^{-1} , $i = 1,2,\cdots,n$ and its largest singular value is σ_n^{-1} . Thus, $|a_{ii}|^{-1} \leq \sigma_n^{-1}$ and now (1) follows. □

Given a nonsingular matrix, $||A||$ (where $||\cdot||$ is 1,2 or ∞ norm) can be easily calculated. The following technique for estimating $||A^{-1}||$ is given in [11]:

<u>Theorem 2</u>. Let A be nonsingular. Let x and y be two vectors such that

$$\begin{aligned} A'x &= b \\ Ay &= x \quad , \end{aligned} \tag{2}$$

where each component of b is choosen to be +1 or -1 so that the solution x is as large as possible. Then,

$$r = ||y|| \;/\; ||x|| \quad , \tag{3}$$

provides an estimate for $||A^{-1}||$. □

An algorithm is given in [11], which employes the QR or the LU decomposition of A to calculate $||A^{-1}||_1$. It involves $O(n^2)$ operations, once the decomposition is known. Also, it is shown experimentally, that this technique provides a reliable indication of the order of the magnitude of condition number. This technique is implemented in LINPACK, a collection of FORTRAN subroutines for solving various forms of linear equations. In [43] it is shown that the estimate r given by (3) is norm dependent. Further in case the LU decomposition of A is known,

$$r_1 = \max(||y||_1 \;/\; ||X||_1 \quad , \quad ||x||_\infty) \quad ,$$

where x and y are given by (2), is shown to give a better estimate of $||A^{-1}||_1$ for matrices of small order. Another technique is given in [18] for estimating $||A^{-1}||_1$.

BIBLIOGRAPHY

[1] Amir-Moéz, A. R., and Fass, A.L.; Elements of Linear Spaces, Pergamon Press, Oxford, 1962.

[2] Bailey, D.W., and Crabtree, D.E.; Bounds for Determinants, Linear Algebra and Appl. 2, 303-309 (1969).

[3] Beesach, P.R.; The Spread of Matrices and Polynomials, Linear Algebra and Appl. 31, 145-149 (1980).

[4] Bellman, R.; Introduction to Matrix Analysis, 2nd ed., McGraw Hill, 1970.

[5] Bellman, R.; The Largest and the Smallest Characteristic Roots of a Positive Definite Matrix, Jour. of Math. Ana. and Applications 90, 217-208 (1982).

[6] Berman, A., and Plemmons, R.J.; Nonnegative Matrices in the Mathematical Sciences, Academic Press, 1979.

[7] Bodewig, E.; Matrix Calculus, Interscience Publishers, 1956.

[8] Brauer, A., and Gentry, I.C.; Bounds for the Greatest Characteristic Root of an Irreducible Nonnegative Matrix, Linear Algebra and Appl. 8, 105-107 (1974).

[9] Brauer, A., and Gentry, I.C.; Bounds for the Greatest Characteristic Root of an Irreducible Nonnegative Matrix II, Linear Algebra and Appl. 13, 109-114 (1976).

[10] Brauer, A., and Mewborn, A.C.; The Greatest Distance Between Two Characteristic Roots of a Matrix, Duke Math. 26, 653-661 (1959).

[11] Cline, A.K., Moler, C.B., Stewart, G.W., and Wilkinson, J.H.; An estimate for the condition number of a matrix, SIAM J. Numer. Anal. 16, 368-375 (1979).

[12] Drazin, M.P., and Haynsworth, E.; Criteria for the reality of a matrix eigenvalue, Math. Zeitschr. 78, 449-452 (1962).

[13] Fan, K., and Hoffman, A.J.; Some metric inequalities in space of matrices, Proc. Amer. Math. Soc. 6, 111-116 (1955).

[14] Goldberg, M, and Zwas, G.; On Matrices Having Equal Spectral Radius and Spectral Norm, Linear Algebra and Appl. 8, 427-434 (1974).

[15] Graybill, F.A.; Matrices with Applications in Statistics, 2nd ed., Wadsworth Publishing Co., 1983.

[16] Grone, R., Johnson, C.R., Sá, E.M., and Wolkowicz, H.; Normal Matrices, Linear Algebra and Appl., to appear.

[17] Grone, R., Johnson, C.R., Sá, E.M., and Wolkowicz, H.; Improving Hadamerd's Inequality, Linear Algebra and Appl., to appear.

[18] Hager, W.W.; Condition Estimates, SIAM J. Sci. Stat. Comput., to appear.

[19] Hall, C.A., and Porsching, T.A.; Bounds for the Maximal Eigenvalue of a Nonnegative Irreducible Matrix, Duke Math. 36, 159-164 (1969).

[20] Henrici, P.; Bounds for iterates, inverses, spectral variation and fields of values of non-normal matrices, Numer. Math. 4, 24-40 (1962).

[21] Holladay, J.C., and Varga, R.S.; On powers of non-negative matrices, Proc. Amer. Math. Soc. 9, 631-634 (1958).

[22] Jennings, A.; Bounds for the Singular Values of a Matrix, IMA Jour. of Numerical Analysis 2, 459-474 (1980).

[23] Johnson, C.R., Kumar, R., and Wolkowicz, H.; Lower Bounds for the Spread of a Matrix, Submitted.

[24] Johnston, R.L., and Smith, B.T.; Calculation of Best Isolated Gerschgorin Discs, Numer. Math. 16, 22-31 (1970).

[25] Kress, R., Vries, M.L., and Wegmann, R.; On Nonnormal Matrices, Linear Algebra and Appl. 8, 109-120 (1974).

[26] Lancaster, P.; Theory of Matrices, Academic Press, 1969.

[27] Lawson, C.L., and Hanson, R.J.; Solving Least Squares Problems, Prentice-Hall, 1974.

[28] Liqun Qi; Some Simple Estimates for Singular Values of a Matrix, Linear Algebra and Appl. 56, 105-119 (1984).

[29] Marcus, M.; An Inequality Connecting the P-condition Number and the Determinant, Numer. Math. 4, 350-353 (1962).

[30] Marcus, M.; Lengths of Tensors in Inequalities I, O. Shisha ed., Academic Press, 1979.

[31] Marcus, M., and Minc, H.; A Survey of Matrix Theory and Matrix Inequalities, Allyn and Bacon, Boston, 1964.

[32] Marshall, A.W., and Olkin, I.; Inequalities: Theory of Majorization and its Applications, Academic Press, 1979.

[33] Medley, H.I.; A Note on a Paper of Johnston and Smith's Concerning Best Isolated Gerschgorin Disks, Numer. Math. 16, 435-441 (1971).

[34] Medley, H.I., and Varga, R.S.; On Smallest Isolated Gerschgorin Disks for Eigenvalues II, Numer. Math. 11, 320-323 (1968).

[35] Medley, H.I., and Varga, R.S.; On Smallest Isolated Gershgorin Disks for Eigenvalues III, Numer. Math. 11, 361-369 (1968).

[36] Merikoski, J.K.; On a Lower Bound for the Perron Eigenvalue , BIT 19, 39-42 (1979).

[37] Merikoski, J.K., Styan, G.P.H., and Wolkowicz, H.; Bounds for Ratios of Eigenvalues Using Traces, Linear Algebra and Appl. 55, 105-124 (1983).

[38] Merikoski, J.K., and Wolkowicz, H.; Improving Eigenvalue Bounds Using Extra Bounds, Linear Algebra and Appl., to appear.

[39] Mirsky, L.; An Introduction to Linear Algebra, Oxford Univ. Press, 1955.

[40] Mirsky, L.; The Spread of a Matrix, Mathematika 3, 127-130 (1956).

[41] Mirsky, L.; Inequalities for Normal and Hermitian Matrices, Duke Math. 24, 591-599 (1957).

[42] Noble, B.; Applied Linear Algebra, Prentice-Hall, 1969.

[43] O'Leary, D.P.; Estimating Matrix Condition Numbers, SIAM J. Sci. Stat. Comput. 1, 205-209 (1980).

[44] Ostrowski, A., and Schneider, M.; Bounds for the Maximal Characteristic Root of a Non-negative Irreducible Matrix, Duke Math. 27, 547-553 (1960).

[45] Parker, W.V.; The Characteristic Roots of Matrices, Duke Math. 12, 519-526 (1945).

[46] Parker, W.V.; Characteristic Roots and the Field of Values of a Matrix, Duke Math. 15, 439-442 (1948).

[47] Parker, W.V.; Sets of Complex Numbers Associated with a Matrix, Duke Math. 15, 711-715 (1948).

[48] Porsching, T.A.; Diagonal Similarity Transformations which Isolate Gerschgorin Discs, Numer. Math. 8, 437-443 (1966).

[49] Ptak, V.; Norms and the spectral radius of matrices, Czech Math. J. 12, 555-557 (1962).

[50] Schneider, H.; An Inequality for Latent Roots Applied to Determinants with Dominant Principal Diagonal, J. London Math. Soc. 28, 8-20 (1953).

[51] Seneta, E.; Non-negative Matrices, An Introduction to Theory and Applications, John Wiley, 1973.

[52] Slepian, D., and Landau, H.; A Note on the Eigenvalues of Hermitian Matrices, SIAM J. Math. Anal. 9, 291-297 (1978).

[53] Stone, B.J.; Best possible ratios of certain matrix norms, Numer. Math. 4, 114-116 (1962).

[54] Taussky, O.; On a Recurring Theorem on Determinants, Amer. Math. Soc. 56, 672-676 (1949).

[55] Thompson, R.C.; Principal Submatrices IX: Interlacing Inequalities for Singular Values of Submatrices, Linear Algebra and Appl. 5, 1-12 (1972).

[56] Thompson, R.C.; Singular Values, Diagonal Elements, and Convexity, SIAM J. Appl. Math. 32, 39-63 (1977).

[57] Todd, John; Survey of Numerical Analysis, McGraw-Hill, 1962.

[58] Tood, John; On Smallest Isolated Gerschgorin Disks for Eigenvalues, Numer. Math. 7, 171-175 (1965).

[59] Varga, R.S.; Matrix Iterative Analysis, Prentice-Hall, 1962.

[60] Varga, R.S.; On Smallest Isolated Gerschgorin Disks for Eigenvalues, Numer. Math. 6, 366-374 (1964).

[61] Wilkinson, J.H.; The Algebraic Eigenvalue Problem, Oxford Univ. Press, 1965.

[62] Wolkowicz, H.; Generating Eigenvalue Bounds Using Optimization, Research Report, Univerity of Alberta.

[63] Wolkowicz, H., and Styan, G.P.H.; Bounds for Eigenvalues Using Traces, Linear Algebra and Appl. 29, 471-506 (1980).

[64] Wolkowicz, H., and Styan, G.P.H.; More Bounds for Eigenvalues Using Traces, Linear Algebra and Appl . 31, 1-17 (1980).